



Draft Model Content Frameworks for Mathematics

Introduction:

Purpose for the Model Content Frameworks for Mathematics

As part of its proposal to the U.S. Department of Education, the Partnership for Assessment of Readiness for College and Careers (PARCC) committed to developing model content frameworks for English language arts/ literacy and mathematics to both serve as a bridge between the standards and the PARCC assessments and provide greater insight into the Common Core State Standards. Therefore, the following pages contain detailed information for grades 3–8 and high school.

The Model Content Frameworks for Mathematics are designed with the following purposes in mind:

- identifying the big ideas in the Common Core State Standards for each grade level,
- helping determine the focus for the various PARCC assessment components, and
- supporting the development of the assessment blueprints.

To ensure strong alignment to the standards, the PARCC Model Content Frameworks were developed through a state-led process between PARCC state content experts and members of the Common Core State Standards writing teams. *The Model Content Frameworks are not a curriculum*; rather, they are a voluntary resource to help teachers understand how to implement the standards. However, given their focus on the big ideas in each grade, the Model Content Frameworks will naturally have relevance for curriculum planning, as well. In addition, teachers may use the frameworks to better understand the standards and how key elements of the assessment design interact with the standards within a grade and across grades.

The following pages contain additional insights into the mathematics standards, followed by grade-by-grade analyses for grades 3–8 and course-based analyses in high school for both an algebra-geometry-algebra sequence and an integrated sequence, as well by conceptual category.

Structure of the Model Mathematics Content Frameworks for Mathematics

The Model Content Frameworks for Mathematics for each grade are written with the expectation that students develop content knowledge, conceptual understanding, and expertise with the Standards for Mathematical Practices. A detailed description of all features of the standards would be significantly lengthier and dense. For that reason, the analyses given here are intended to be valuable starting points. The Model Content Frameworks for mathematics provide guidance for grades 3–8 and high school across a number of areas:

Examples of key advances from the previous grade: This component highlights some of the major grade-to-grade steps in the progression of increasing knowledge and skill detailed in the standards. Note that each key advance in mathematical content also corresponds to a widening scope of problems that students can solve. Examples of key advances are highlighted to stress the need to treat topics in ways that take into account where students have been in previous grades and where they will be going in subsequent grades.¹

¹ See the *Progressions* documents for additional information about progressions in the standards.
<http://commoncoretools.wordpress.com/>

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Fluency expectations and examples of culminating standards: This section highlights individual standards that set expectations for fluency or that otherwise represent culminating masteries. These standards are highlighted to stress the need to provide sufficient supports and opportunities for practice to help students meet these expectations. Fluency is not meant to come at the expense of understanding but is an outcome of a progression of learning and sufficient thoughtful practice. It is important to provide the conceptual building blocks that develop understanding in tandem with skill along the way to fluency; the roots of this conceptual understanding often extend one or more grades earlier in the standards than the grade when fluency is finally expected.

Examples of major within-grade dependencies: This piece highlights cases in which a body of content in a given grade depends conceptually or logically upon another body of content within that same grade. Examples of within-grade dependencies are highlighted to stress the need to organize material coherently within each grade level. (Because of space limitations, only examples of large-scale dependencies are described here, but coherence is important for dependencies that exist at finer grain size as well.)

Examples of opportunities for connections among standards, clusters, or domains: This component highlights opportunities for connecting content in assessments, as well as in curriculum and instruction. Examples of connections are highlighted to stress the need to avoid approaching the standards as merely a checklist.

Examples of opportunities for in-depth focus: This section highlights some individual standards that play an important role in the content at each grade level. The indicated mathematics might be given an especially in-depth treatment, as measured by, for example, the type of assessment items; the number of days; the quality of classroom activities to support varied methods, reasoning, and explanation; the amount of student practice; and the rigor of expectations for depth of understanding or mastery of skills.²

Examples of opportunities for connecting mathematical content and mathematical practices: This section provides some examples of how students may engage in the mathematical practices as they learn the mathematics of the grade.³ These examples are provided to stress the need to connect content and practices, as required by the standards. In addition to the concrete examples provided in each grade, the following are some general comments about connecting content and practices:

- Connecting content and practices happens in the context of **working on problems**. The very first Standard for Mathematical Practice is to make sense of problems and persevere in solving them (MP.1).
- Particularly in grades K–8, making sense of problems (MP.1) involves the pervasive use of **visual representations as tools** (MP.5) for understanding and explaining computation and problem solving (MP.2, 6).
- As the above point suggests, the Standards for Mathematical Practice interact and overlap with each other. **They are not a checklist.**

² Note, however, that a standard can be individually important even though the indicated mathematics may require relatively little teaching time.

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Instructional emphases by cluster: The four or five domains in each grade contain anywhere from nine to 12 clusters. Some clusters require greater emphasis than others. At the end of each grade analysis, a table shows the relative emphasis for each cluster. **Prioritization does not imply neglect of material.** No material in the standards should be excluded. But clear priorities can help ensure that the relative importance of content is properly attended to. Note that the prioritization is in terms of cluster headings. Setting priorities at the cluster level is a way to talk reasonably specifically about mathematical content while still maintaining the coherence of the mathematics at each grade.

Additionally, background guidance around the development of materials, including guidance on better understanding the standards themselves, is included in the Model Content Frameworks as an introduction to each grade level. The background guidance and the grade-level guidance are intended to be used *together* to provide educators with a substantially deeper understanding of the standards themselves.

Connections to Assessment

The PARCC Assessment System will be designed to measure the knowledge, skills, and understanding essential to achieving college and career readiness. In mathematics, this includes conceptual understanding, procedural skill and fluency, and application and problem solving, as defined by the standards. Each of these works in conjunction with the others to promote students' achievement in mathematics. To measure the full range of the standards, the assessments will include tasks that require students to connect mathematical content and mathematical practices.

The Model Content Frameworks for Mathematics reflect these priorities by providing detailed information about selected practice standards, fluencies, connections, and priorities organized at the cluster level.

With the recent refinements made to the PARCC Assessment System (see <http://tinyurl.com/PARCCletter62411> for more information), the Model Content Frameworks do not contain a suggested scope and sequence by quarter. Rather, they provide examples of key content dependencies (where one concept ought to come before another), key instructional emphases, opportunities for in-depth work on key concepts, and connections to critical practices. These last two components, in particular, intend to support local and state curriculum developers' efforts to connect content and practices and achieve the standards' balance of conceptual understanding and procedural fluency.

Overall, as proposed, the PARCC Assessment System would include a mix of short- and extended-response items, performance tasks, and computer-based selected-response items. In mathematics, the assessment system is designed to measure students' understanding of key big ideas indicated in the standards, with emphasis given to both the content standards and the practice standards. Questions asked would measure student learning across various mathematical domains and practices. The variety of the questions will attend to the full range of mathematics, including conceptual understanding, procedural fluency, and the varieties of expertise described by the practice standards. In short, mathematical understanding and procedural skill are equally important, and both can be assessed using mathematical tasks of sufficient richness, which PARCC will include in its assessment system.



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Principles Regarding the Common Core State Standards for Mathematics

Focus and Coherence

The two major evidence-based principles that support the standards are **focus** and **coherence**. **Focus** is necessary so that students have sufficient time to think, practice, and integrate new ideas into their growing knowledge structure. Focus is also a way to allow time for the kinds of rich classroom discussion and interaction that support the Standards for Mathematical Practices.

Giving students and teachers the time they need to focus means that all of the standards are not of equal priority. The amount and intensity of instructional time spent working with different mathematical content will also vary depending upon the scope and depth of each standard and whether the content is new material or has been introduced to the student previously. Note, however, that all standards should be addressed, as they all play a role in students' mathematical development. Assessments, too, will reflect patterns of content emphasis and de-emphasis in the standards and will measure core content in depth.

The second principle, **coherence**, arises from mathematical connections. Some of the connections in the standards knit topics together at a single grade level (such as area models and multiplication in grade 3). Most connections, however, play out across two or more grade levels to form a progression of increasing knowledge, skill, or sophistication. The standards are not so much assembled out of topics as they are woven of these progressions. Likewise, instruction at any given grade would benefit from being informed by a sense of the overall progression students are following across the grades.

Another set of connections is found between the content standards and the practice standards. These connections are absolutely essential to support the development of students' broader mathematical understanding. To reflect the standards, the Model Content Frameworks emphasize that mathematics is not a checklist of fragments to be mastered, but that doing and using mathematics involves connecting content and practices.

Focus is critical to ensure that students learn the most important content completely, rather than succumbing to an overly broad survey of content. Coherence is critical to ensure that students see mathematics as a logically progressing discipline, which has intricate connections among its various domains and requires a sustained practice to master. Focus shifts over time, as seen in the following:

- In grades K–5, the focus is on the addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals, with a balance of concepts, skills, and problem solving. Arithmetic is viewed as an important set of skills and also as a thinking subject that, done thoughtfully, prepares students for algebra. Measurement and geometry develop alongside number and operations and are tied specifically to arithmetic along the way.
- In middle school, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as arithmetic matures into algebra. The theme of quantitative relationships also becomes explicit in grades 6–8, developing into the formal notion of a function by grade 8. Meanwhile, the foundations of high school deductive geometry are laid in the middle grades. Finally, the gradual development of data representations in K–5 leads to



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statistics with the study of distributions, central tendency, variation, and association in middle school.

- In high school, algebra, functions, geometry, and statistics develop with an emphasis on modeling. Students continue to take a thinking approach to algebra, learning to see and make use of structure in algebraic expressions of growing complexity. As this description suggests, mathematical content in all grades is best approached in the ways envisioned by the Standards for Mathematical Practice.

Viewing the Standards through Four Lenses

While many views of the standards provide viable pieces of information, four lenses in particular provide views of the entire landscape of the standards: fluency, content, practice, and priority. By carefully considering each, educators can maximize the development of their materials.

Fluency Lens

At each grade level in the standards, one or two fluencies are expected:

Table 1: Grade-Level Fluency

Grade	Required Fluency
K	Add/subtract within 5
1	Add/subtract within 10
2	Add/subtract within 20
	Add/subtract within 100 (pencil and paper)
3	Multiply/divide within 100
	Add/subtract within 1,000
4	Add/subtract within 1,000,000
5	Multidigit multiplication
6	Multidigit division
	Multidigit decimal operations
7	Solve $px + q = r$, $p(x + q) = r$

Fluent in the standards means “fast and accurate.” It might also help to think of fluency as meaning more or less the same as when someone is said to be fluent in a foreign language. To be fluent is to flow; fluent isn’t halting, stumbling, or reversing oneself.

The word *fluency* was used judiciously in the standards to mark the endpoints of progressions of learning that begin with a solid foundation and then pass upward through stages of growing maturity. The rarity of the word itself might easily lead to fluency becoming invisible in the standards — one more among 25 things in a grade, easily overlooked. But many standards in prior grades work together to contribute to fluency, and those parts of the standards can be better understood in relation to the goal of fluency. Paths leading to fluency are thus one of the organizing principles of the standards. For example, fluency with single-digit addition (2.OA.2) is a major progression of learning stretching back in the standards to kindergarten. This is because a key aspect of fluency is that it is not something that happens all at once in a single grade, but requires attention to student understanding along the way. It



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is important to ensure that sufficient practice (and, if necessary, extra support) are provided at each grade to allow all students to reach fluency.

Content Lens and Practice Lens Connected

The standards as a whole contain Standards for Mathematical Content and Standards for Mathematical Practice. These are meant to be connected, as noted in the Common Core State Standards for Mathematics (page 8):

“Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.”

The word *connect* in this passage is important. Separating the practices from the content is not helpful and is not what the standards require. The practices do not exist in isolation; the vehicle for engaging in the practices is mathematical content.

The Standards for Mathematical Practice should be embedded in classroom instruction, discussions, and activities. Moreover, the practices should not be viewed as a checklist of separate questions or tasks to be added to lessons. They describe the kind of mathematics teaching and learning to be fostered in the classroom. Grade-appropriate mathematical reasoning and discussions about such reasoning are the core of the mathematical practices. To promote such an environment, students should have opportunities to work on carefully designed, standards-based mathematical tasks that can vary in difficulty, context, and type. Carefully designed standards-based mathematical tasks will reveal students' content knowledge and elicit evidence of mathematical practices. For example, a mathematical task keyed to a content standard about solving equations might maximize opportunities to “recognize and make use of structure” in solving the equation. Mathematical tasks are thus an important opportunity to connect content and practices. Any given mathematical task might be designed to elicit and develop the varieties of expertise described in the Standards for Mathematical Practice. To be consistent with the standards as a whole, assessment as well as curriculum and classroom activities must include a balance of mathematical tasks that provide opportunities for students to develop the kinds of expertise described in the practices. The requirement of attending to the mathematical practices is one of the general criteria noted above for the development of mathematics curriculum.

To align with strong instruction, PARCC assessments will include several types of tasks. The task types will integrate the content standards and the Standards for Mathematical Practice. Task types will include shorter items and longer, constructed items and will vary in technical difficulty. While they are not part of this document, PARCC is currently developing prototype tasks that will be made available in the near future.

Priority Lens

The standards focus on crucial material so that students can have more time to discuss, reflect upon, and practice it. The standards treat mathematics as a coherent subject to promote the sense-making that fuels mastery. The principles of focus and coherence are the twin engines that must be carried forward in implementation efforts and substantiated in curricula and assessments. And while mathematics education is about more than mathematical content, everything in mathematics education



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ultimately happens through mathematical content. Therefore it is necessary to prioritize within that content.

The actual demands of college and careers mean mastering some material in the standards is more important than mastering other material. To keep as many students as possible on the path to readiness for college and careers, students need to be given enough time to succeed in these areas. The Model Content Frameworks for Mathematics recommend a prioritization of content in the standards for purposes of focusing time, effort, and investment.

The prioritizations seen in the Model Content Frameworks were drawn, in part, from the critical areas in the standards. While the critical areas provide a general overview of major topics in a year, the priorities in the Model Content Frameworks are written at the detail of the cluster. Thus they do not replicate the critical areas themselves.

Given the design of the content standards — and given what they actually say — the best way to balance forces is to work primarily at the cluster level. Thus, each grade level's priorities are clearly identified in accompanying tables.

Starting Points for Transition to the Common Core State Standards

Identified in this section are a few particularly rich areas of mathematical content that can be used by assessment designers, teachers, principals, state and district staff members, and teacher educators as starting points to coordinate and concentrate efforts to transition to the standards. Special attention should be given to how well current materials treat these areas. Organizing implementation work according to progressions is recommended because the instructional approach to any given topic should be informed by its place in an overall flow of ideas. Many of these same areas are the focus of the item prototyping currently underway as part of the development of the PARCC Assessment System.

Please note that particular mathematical practices with which to begin are not listed because doing so may unintentionally lead to a misunderstanding of the nature of mathematical practice itself. The mathematical practices are not a to-do list, nor are they like filing cabinets that one can sort behaviors into. When a student working on a real-world geometry problem in class questions whether another student's drawing is precise enough, the question involves issues of precision as well as modeling, not to mention communication and argument. In short, a single classroom question or behavior might reflect several practices at once.

The following suggestions are not meant to reorganize the standards into a new structure. In fact, a glance will show that the list is incomplete. By providing a focused list of suggested starting points, the risk of taking on too much and doing none of it well is minimized.

- Counting and Cardinality and Operations and Algebraic Thinking (particularly in the development of an understanding of quantity): grades K–2.
- Operations and Algebraic Thinking: multiplication and division in grades 3–5, tracing the evolving meaning of multiplication, from equal groups and array/area thinking in grade 3 to all multiplication situations in grade 4 (including multiplicative comparisons) and from whole numbers in grade 3 to decimals and fractions in grades 5 and 6.
- Number and Operations — Base Ten: addition and subtraction in grades 1–4.



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- Number and Operations — Base Ten: multiplication and division in grades 3–6.
- Number and Operations — Fractions: fraction addition and subtraction in grades 4–5, including parallel development of fraction equivalence in grades 3–5.
- Number and Operations — Fractions: fraction multiplication and division in grades 4–6.
- The Number System: grades 6–7.
- Expressions and Equations: grades 6–8, including how this extends prior work in arithmetic.
- Ratio and Proportional Reasoning: its development in grades 6–7, its relationship to functional thinking in grades 6–8, and its connection to lines and linear equations in grade 8.
- Geometry: work with the coordinate plane in grades 5–8, including connections to ratio, proportion, algebra, and functions in grades 6–high school.
- Geometry: congruence and similarity of figures in grades 8–high school, with emphasis on real-world and mathematical problems involving scales and connections to ratio and proportion.
- Modeling: focused on equations and inequalities in high school, development from simple modeling tasks such as word problems to richer more open-ended modeling tasks.
- Seeing Structure in Expressions: from expressions appropriate to grades 8–9 to expressions appropriate to grades 10–11.
- Statistics and Probability: comparing populations and drawing inferences in grades 6–high school.
- Units as a cross-cutting theme in the areas of measurement, geometric measurement, base-ten arithmetic, unit fractions, and fraction arithmetic, including the role of the number line.

Many of these stressed areas are likely to be glossed over as “something that is already in the curriculum” — yet the standards require more. As noted in the standards, these or any content areas are best approached in the ways envisioned by the Standards for Mathematical Practice. The reason for greater focus is to give students and teachers more time — time to discuss, reason with, reflect upon, and practice mathematics. These identified areas of mathematics are sufficiently rich to allow the mathematical practices to come alive.

The standards are a challenging vision for higher mathematics performance. By suggesting starting points, the aim is, in part, to define some content boundaries to help focus the innovation in the creation of new materials and to drive innovation in assessment items.

Using the Model Content Frameworks to Support All Students

It is critical that all students are able to access the standards and demonstrate mastery of the skills and knowledge embedded within them. The Model Content Frameworks are written to support the use of the principles of universal design for learning (UDL). UDL “recommends ways to provide cognitive as well as physical access to the curriculum.”⁴ UDL offers assessment designers and educators a research-based blueprint for designing materials that accommodate individual differences. Materials designed based on the Model Content Frameworks and that use UDL offer flexible learning environments appropriate for

⁴ Center for Applied Special Technology (CAST). www.cast.org/index.html



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all students to accommodate individual learning differences.⁵ To ensure that a student's entire learning experience is aligned to those high expectations, the PARCC Assessment System will be designed and developed using principles of universal design. Just as the standards should be accessible to all students, so should the assessments that align to the standards.

Guidance Regarding the Use of Resources in Mathematics

In the early phases of materials development, it is wise to consider the degree to which currently used materials align to the standards. In this process, however, it is critical to note that the content standards are statements — not topics. Coverage of topics is therefore not a guarantee of alignment, and it may affect alignment negatively when the coverage is wide and/or shallow. Cluster headings often unify the standards in the cluster by communicating their joint intent. Aligning to the standards requires taking into account the guidance to be gained from cluster headings, grade-level introductions, indicators of opportunities for modeling or use of an applied approach, and so forth. In the context of a multi-grade progression, alignment also means treating the content in ways that take into account the previous stage of the progression and anticipate the next.

At the heart of the Model Content Frameworks for Mathematics is a commitment to provide educators with guidance on the implementation of the Common Core State Standards, particularly with respect to the needs of states and districts as they develop, obtain, or revise materials to meet the standards. Therefore, a number of important criteria are suggested for the development of additional curricular or instructional materials or for reviewing existing resources.

- Materials should help students meet the indicated Standards for Mathematical Content. Materials must also equip teachers and students to develop the varieties of expertise described in the Standards for Mathematical Practice.
- Materials should be mathematically correct. The beauty and applied power of the subject should be evident. Materials should be engaging for a diverse body of students, and this engagement must exist side by side with the practice and hard thinking that is often necessary for learning mathematics.
- Materials that are excellent but narrow in scope still have value; they can be combined with other like resources and supplemented as necessary. This is better than settling for a single mediocre resource that claims to cover all content.
- Materials should reflect the standards' balanced approach to mathematics, connecting content and practices and achieving a balance of conceptual understanding and procedural skill and fluency. Specific aspects of achieving this balance include:
 - Balance of tasks and activities:* Activities, tasks, and problems for students must exhibit balance along various dimensions. For example, some activities and tasks can call for procedural skill and fluency alone, others can call for conceptual understanding, and still others should require skill and understanding in equal measure. Some can be brief practice exercises; others can require longer chains of reasoning. Some can be abstract; others can be contextual. Well-chosen tasks

⁵ Please see the CAST website for suggestions for how to differentiate learning for students with disabilities. The CAST website can be accessed at www.cast.org/index.html.



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can also demonstrate the importance of mathematics in daily living for students, including connecting to other areas of students' interest, such as population growth and history, data and sports, and financial decision making.

Balance in how time is spent: There should be time for whole-class or group discussion and debate, time for solitary problem solving and reflection, and time for thoughtful practice and routine skill building. Individual problem solving and explanation of mathematical thinking may be intertwined several times during a class.

Common sense in achieving balance: Not every task, activity, or workweek has to be balanced in these ways. It is reasonable to have phases of narrow intensity, during which tasks, activities, and time are concentrated in a single mode.

- Materials should draw the teacher's attention explicitly to nuances in the content being addressed and to specific opportunities for teachers to foster mathematical practices in the study of that content.
- Materials must give teachers workable strategies for helping students who have special needs, such as students with disabilities, English language learners, and gifted students.
- Materials must give teachers strategies that help build upon and develop students' literacy skills, including reading, writing, speaking, and listening (as seen in the English language arts/literacy standards) to allow students to deepen their understanding of mathematics concepts. Materials must help students acquire comprehension skills and strategies, such as helping students understand the meanings of specialized vocabulary, symbols, units, and the variety of expressions of mathematics and data to support students' growth in attending to precision. Materials should provide teachers with strategies to give students opportunities to engage in mathematical discourse using both informal language and precise language to convey ideas, communicate solutions, and support arguments.

Additional Resources

Members of the working group and writing team for the *Common Core State Standards for Mathematics* are developing some resources to inform the development of curriculum and instruction aligned to the standards.

PARCC Resources

In the future, PARCC intends to build additional supplementary material to further illustrate implementation of the standards, which may include model instructional units and sample tasks. In general, they will likely focus on areas of the standards that are particularly new to educators to support transition efforts. For example, the reader will note that grade 8 of the Model Content Frameworks includes a key opportunity for such work to occur around linear equations, the geometry of lines, and proportional reasoning. As these become available, they will be published for voluntary use at <http://parcconline.org>.

Progressions

The progressions are being developed by members of the Common Core State Standards working group and writing team through the University of Arizona's Institute for Mathematics and Education. Progressions are narratives of the standards that describe how student skill and understanding in a



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particular domain develop from grade to grade. One of the primary uses of the progressions is to give educators and curriculum developers information that can help them develop materials for instruction aligned to the standards. <http://commoncoretools.wordpress.com/>

Illustrative Mathematics

Under the guidance of members of the working group as well as other national experts in mathematics and mathematics education, The Illustrative Mathematics Project will illustrate the range and types of mathematical work that students will experience in a faithful implementation of the Common Core State Standards and by publishing other tools that support implementation of the standards.

<http://illustrativemathematics.org>

Common Core Tools

Additional tools that continue to be developed are posted from time to time on <http://commoncoretools.wordpress.com>, a blog moderated by Dr. William McCallum, distinguished professor and head of mathematics at the University of Arizona and mathematics lead for the *Common Core State Standards for Mathematics*.



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Grade-by-Grade Standards Analyses

The following pages provide insights into the standards for grades 3–8. For each grade, analysis is provided in the following categories:

Examples of Key Advances from the Previous Grade

- Highlights some of the major grade-to-grade steps in the progression of increasing knowledge and skill detailed in the standards. Note that each key advance in mathematical content also corresponds to a widening scope of problems that students can solve. Examples of key advances are highlighted to stress the need to treat topics in ways that take into account where students have been in previous grades and where they will be going in subsequent grades.¹

Fluency Expectations and Examples of Culminating Standards

- Highlights individual standards that set expectations for fluency or that otherwise represent culminating masteries. These standards are highlighted to stress the need to provide sufficient supports and opportunities for practice to help students meet these expectations. Fluency is not meant to come at the expense of understanding but is an outcome of a progression of learning and sufficient thoughtful practice. It is important to provide the conceptual building blocks that develop understanding in tandem with skill along the way to fluency; the roots of this conceptual understanding often extend one or more grades earlier in the standards than the grade when fluency is finally expected.

Examples of Major Within-Grade Dependencies

- Highlights cases in which a body of content within a given grade depends conceptually or logically upon another body of content within that same grade. Examples of within-grade dependencies are highlighted to stress the need to organize material coherently within each given grade. (Because of space limitations, only examples of large-scale dependencies are described here, but coherence is important for dependencies that exist at finer grain size as well.)

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Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- Highlights opportunities for connecting content in assessments, as well as in curriculum and instruction. Examples of connections are highlighted to stress the need to avoid approaching the standards as merely a checklist.

Examples of Opportunities for In-Depth Focus

- Highlights some individual standards that play an important role in the content at each grade. The indicated mathematics might be given an especially in-depth treatment, as measured, for example, by the type of assessment items; the number of days; the quality of classroom activities to support varied methods, reasoning, and explanation; the amount of student practice; and the rigor of expectations for depth of understanding or mastery of skills.²

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- Provides some examples of how students may engage in the mathematical practices as they learn the mathematics of the grade.³ These examples are provided to stress the need to connect content and practices, as required by the standards.
- In addition to the concrete examples provided in each grade, the following are some general comments about connecting content and practices:
 - Connecting content and practices happens in the context of **working on problems**. The very first Standard for Mathematical Practice is to make sense of problems and persevere in solving them (MP.1).
 - Particularly in grades K–8, making sense of problems (MP.1) involves the pervasive use of **visual representations as tools** (MP.5) for understanding and explaining computation and problem solving (MP.2, 6).
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Instructional Emphases by Cluster

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Please Note

- The words *examples* and *opportunities* in the above categories emphasize that the analysis provided in each category is not exhaustive. For example, there are many opportunities to connect mathematical content and practices in every grade, there are many opportunities for in-depth focus in every grade, and so on. A comprehensive description of these features of the standards would be hundreds of pages long. ***The analyses given here should be thought of as valuable starting points.***
- Always refer back to the *Common Core State Standards for Mathematics* for exact language about student expectations.



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Grade 3 Standards Analysis

Examples of Key Advances from Grade 2 to Grade 3

- Students in grades K–2 worked on addition and subtraction concepts, skills, and problem solving. Beginning in grade 3, students will learn concepts, skills, and problem solving for multiplication and division. This work will continue in grades 3, 4, and 5, preparing the way for work with ratios and proportional relationships in grades 6 and 7.
- Students in grade 3 also begin to enlarge their concept of number by developing an understanding of fractions as numbers. This work will continue in grades 3–6, preparing the way for work with the rational number system in grades 6 and 7.

Fluency Expectations and Examples of Culminating Standards

- 3.OA.7** Students fluently multiply two single-digit factors, and they fluently find related quotients. By the end of grade 3, they know single-digit products from memory.
- 3.NBT.2** Students fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. (Although 3.OA.7 and 3.NBT.2 are both fluency standards, these two standards do not represent equal investments of time in grade 3. Note that students in grade 2 were already adding and subtracting within 1000, just not fluently. That makes 3.NBT.2 a relatively small and incremental expectation. By contrast, multiplication and division are new in grade 3, and meeting the multiplication and division fluency standard 3.OA.7 with understanding is a major portion of students' work in grade 3.)

Examples of Major Within-Grade Dependencies

- Students must begin work with multiplication and division (3.OA) at or near the very start of the year to allow time for understanding and fluency to develop. Note that area models for products are an important part of this process (3.MD.7). Hence, work on concepts of area (3.MD.5–6) should likely begin at or near the start of the year as well.

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Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- Students' work with partitioning shapes (3.G.2) relates to visual fraction models (3.NF).
- Scaled picture graphs and scaled bar graphs (3.MD.3) can be a visually appealing context for solving multiplication and division problems.

Examples of Opportunities for In-Depth Focus

- 3.OA.3 Word problems involving equal groups, arrays, and measurement quantities can be used to build students' understanding of and skill with multiplication and division, as well as to allow students to demonstrate their understanding of and skill with these operations.
- 3.OA.7 Finding single-digit products and related quotients is a required fluency for grade 3. Reaching fluency will take much of the year for many students. These skills and the understandings that support them are crucial; students will rely on them for years to come as they learn to multiply and divide with multidigit whole numbers and to add, subtract, multiply, and divide with fractions. After multiplication and division situations have been established, reasoning about patterns in products (e.g., products involving factors of 5 or 9) can help students remember particular products and quotients. Practice — and if necessary, extra support — should continue all year for those who need it to attain fluency.
- 3.MD.2 Continuous measurement quantities such as liquid volume, mass, and so on are an important context for fraction arithmetic (cf. 4.NF.4c, 5.NF.7c, 5.NF.3). In grade 3, students begin to get a feel for continuous measurement quantities and solve whole-number problems involving such quantities.
- 3.MD.7 Area is a major concept within measurement, and area models must function as a support for multiplicative reasoning in grade 3 and beyond.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- Students learn and use strategies for finding products and quotients that are based on the properties of operations; for example, to find 4×7 they may recognize that $7 = 5 + 2$ and compute $4 \times 5 + 4 \times 2$. This is an example of seeing and making use of structure (MP.7). Such reasoning processes amount to brief arguments that students may construct and critique (MP.3).

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- Students will analyze a number of situation types for multiplication and division, including arrays and measurement contexts. Extending their understanding of multiplication and division to these situations requires that they make sense of problems and persevere in solving them (MP.1).

Instructional Emphases by Cluster

The five domains in grade 3 contain 11 clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
Represent and solve problems involving multiplication and division.	Use place value understanding and properties of operations to perform multi-digit arithmetic. ⁵	Represent and interpret data. <i>(Opportunity to link to multiplication and division problem solving.)</i>
Understand the properties of multiplication and the relationship between multiplication and division.	Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.	Reason with shapes and their attributes.
Multiply and divide within 100. ⁴		
Solve problems involving the four operations, and identify and explain patterns in arithmetic.		
Develop understanding of fractions as numbers.		
Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.		
Geometric measurement: understand concepts of area and relate area to multiplication and to addition.		

⁴ Cluster contains a fluency standard.

⁵ Cluster contains a fluency standard.

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Grade 4 Standards Analysis

Examples of Key Advances from Grade 3 to Grade 4

- In grade 3, students studied multiplication in terms of equal groups, arrays, and area. In grade 4, students extend their concept of multiplication to make multiplicative comparisons (4.OA.1).⁶
- Students in grade 4 apply and extend their understanding of the meanings and properties of addition and subtraction to extend addition and subtraction to fractions (4.NF.3).⁷
- Fraction equivalence is an important theme within the standards that begins in grade 3. In grade 4, students extend their understanding of fraction equivalence to the general case, $\frac{a}{b} = \frac{(n \times a)}{(n \times b)}$ (3.NF.3 \rightarrow 4.NF.1).⁸ They apply this understanding to compare fractions in the general case (3.NF.3d \rightarrow 4.NF.2).
- Students in grade 4 apply and extend their understanding of the meanings and properties of multiplication to multiply a fraction by a whole number (4.NF.4).
- Students in grade 4 begin using the four operations to solve word problems involving continuous measurement quantities such as liquid volume, mass, and time (4.MD.2).
- Students combine their understanding of the meanings and properties of multiplication and division with their understanding of base-ten units to begin to multiply and divide multidigit numbers (4.NBT.5–6; this builds on work done in grade 3, cf. 3.NBT.3).
- Students generalize their previous understanding of place value for multidigit whole numbers (4.NBT.1–3). This supports their work in multidigit multiplication and division, carrying forward into grade 5, when students will extend place value to decimals.

⁶ In an additive comparison problem (grades 1–2), the underlying question is *what amount would be added to one quantity to result in the other*. In a multiplicative comparison problem, the underlying question is *what factor would multiply one quantity to result in the other*.

⁷ This work is limited to like denominators in grade 4 to give students more time to build their understanding of fraction equivalence before adding and subtracting unlike denominators in grade 5.

⁸ Students who can generate equivalent fractions can also develop strategies for adding fractions with unlike denominators, but this is not a requirement in grade 4.

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Fluency Expectations and Examples of Culminating Standards

- 4.NBT.4 Students fluently add and subtract multidigit whole numbers using the standard algorithm.

Examples of Major Within-Grade Dependencies

- Students' work with decimals (4.NF.5–7) depends to some extent on concepts of fraction equivalence and elements of fraction arithmetic.
- Standard 4.MD.2 refers to using the four operations to solve word problems involving continuous measurement quantities such as liquid volume, mass, time, and so on. Some parts of this standard could be met earlier in the year (such as using whole-number multiplication to express measurements given in a larger unit in terms of a smaller unit — see also 4.MD.1), while others might be met only by the end of the year (such as word problems involving addition and subtraction of fractions or multiplication of a fraction by a whole number — see also 4.NF.3d and 4.NF.4c).
- Standard 4.MD.7 refers to word problems involving unknown angle measures. Before this standard can be met, students must understand concepts of angle measure (4.MD.5) and, presumably, gain some experience measuring angles (4.MD.6). Before that can happen, students must have some familiarity with the geometric terms that are used to define angles as geometric shapes (4.G.1).

Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- The work that students do with units of measure (4.MD.1–2) and with multiplication of a fraction by a whole number (4.NF.4) can be connected to the idea of “times as much” in multiplication (4.OA.1).
- Addition of fractions (4.NF.3) and multiplication of a fraction by a whole number (4.NF.4) can be connected by the distributive property. Abstractly,

$$3 \times \frac{1}{5} = (1 + 1 + 1) \times \frac{1}{5} = 1 \times \frac{1}{5} + 1 \times \frac{1}{5} + 1 \times \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}.$$

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Examples of Opportunities for In-Depth Focus

- 4.NBT.5 When students work toward meeting this standard, they combine prior understanding of multiplication with deepening understanding of the base-ten system of units to express the product of two multidigit numbers as another multidigit number. This work will continue in grade 5 and culminate in fluency with the standard algorithms in grade 6.
- 4.NBT.6 When students work toward meeting this standard, they combine prior understanding of multiplication and division with deepening understanding of the base-ten system of units to find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors. This work will develop further in grade 5 and culminate in fluency with the standard algorithms in grade 6.
- 4.NF.1 Extending fraction equivalence to the general case is necessary to extend arithmetic from whole numbers to fractions and decimals.
- 4.NF.3 This standard represents an important step in the multigrade progression for addition and subtraction of fractions. Students extend their prior understanding of addition and subtraction to add and subtract fractions with like denominators by thinking of adding or subtracting so many unit fractions.
- 4.NF.4 This standard represents an important step in the multigrade progression for multiplication and division of fractions. Students extend their developing understanding of multiplication to multiply a fraction by a whole number.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- When students decompose numbers into sums of multiples of base-ten units to multiply them (4.NBT.5), they are seeing and making use of structure (MP.7).
- To compute and interpret remainders in word problems (4.OA.3), students must reason abstractly and quantitatively (MP.2).



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Instructional Emphases by Cluster

The five domains in grade 4 contain 12 clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
Use the four operations with whole numbers to solve problems.	Gain familiarity with factors and multiples.	Generate and analyze patterns.
Generalize place value understanding for multi-digit whole numbers.	Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.	Represent and interpret data.
Use place value understanding and properties of operations to perform multi-digit arithmetic. ⁹	Geometric measurement: understand concepts of angle and measure angles.	
Extend understanding of fraction equivalence and ordering.	Draw and identify lines and angles, and classify shapes by properties of their lines and angles.	
Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.		
Understand decimal notation for fractions, and compare decimal fractions.		

⁹ Cluster contains a fluency standard.

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Grade 5 Standards Analysis

Examples of Key Advances from Grade 4 to Grade 5

- In grade 5, students will integrate decimal fractions into the place value system (5.NBT.1–4). By thinking about decimals as sums of multiples of base-ten units, students begin to extend algorithms for multidigit operations to decimals (5.NBT.7).
- Students use their understanding of fraction equivalence and their skill in generating equivalent fractions as a strategy to add and subtract fractions, including fractions with unlike denominators.
- Students apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction (5.NF.4). They also learn the relationship between fractions and division, allowing them to divide any whole number by any nonzero whole number and express the answer in the form of a fraction or mixed number (5.NF.3). And they apply and extend previous understandings of multiplication and division to divide a unit fraction by a whole number or a whole number by a unit fraction.¹⁰
- Students extend their grade 4 work in finding whole-number quotients and remainders to the case of two-digit divisors (5.NBT.6).
- Students continue their work in geometric measurement by working with volume as an attribute of solid figures and as a measurement quantity (5.MD.3–5).
- Students build on their previous work with number lines to use two perpendicular number lines to define a coordinate system (5.G.1–2).

Fluency Expectations and Examples of Culminating Standards

5.NBT.5 Students fluently multiply multidigit whole numbers using the standard algorithm.

Examples of Major Within-Grade Dependencies

¹⁰ Students able to multiply fractions in general can develop strategies to divide fractions in general by reasoning about the relationship between multiplication and division. But the division of a fraction by a fraction is not a requirement at this grade.

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- Understanding that in a multidigit number, a digit in one place represents $\frac{1}{10}$ of what it represents in the place to its left (5.NBT.1) is an example of multiplying a quantity by a fraction (5.NF.4).

Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- The work that students do in multiplying fractions extends their understanding of the operation of multiplication. For example, to multiply $\frac{a}{b} \times q$ (where q is a whole number or a fraction), students must interpret $\frac{a}{b} \times q$ as meaning a parts of a partition of q into b equal parts (5.NF.4a). This interpretation of the product leads to a product that is less than, equal to, or greater than q depending on whether $\frac{a}{b} < 1$, $\frac{a}{b} = 1$, or $\frac{a}{b} > 1$, respectively (5.NF.5).

Examples of Opportunities for In-Depth Focus

- 5.NBT.1 The extension of the place value system from whole numbers to decimals is a major intellectual accomplishment involving understanding and skill with base-ten units and fractions.
- 5.NBT.6 The extension from one-digit divisors to two-digit divisors requires care. This is a major milestone along the way to reaching fluency with the standard algorithm in grade 6 (6.NS.2).
- 5.NF.2 When students meet this standard, they bring together the threads of fraction equivalence (grades 3–5) and addition and subtraction (grades K–4) to fully extend addition and subtraction to fractions.
- 5.NF.4 When students meet this standard, they fully extend multiplication to fractions, making division of fractions in grade 6 (6.NS.1) a near target.
- 5.MD.5 Students work with volume as an attribute of a solid figure and as a measurement quantity. Students also relate volume to multiplication and addition. This work begins a progression leading to valuable skills in geometric measurement in middle school.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

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- When students break divisors and dividends into sums of multiples of base-ten units (5.NBT.6), they are seeing and making use of structure (MP.7). To the extent that multidigit division problems take time and effort, they can also require perseverance (MP.1).
- When students explain patterns in the number of zeros of the product when multiplying a number by powers of 10 (5.NBT.2), they have an opportunity to look for and express regularity in repeated reasoning (MP.8).
- When students show that the volume of a right rectangular prism is the same as would be found by multiplying the side lengths (5.MD.5), they also have an opportunity to look for and express regularity in repeated reasoning (MP.8).

Instructional Emphases by Cluster

The five domains in grade 5 contain 11 clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
Understand the place value system.	Graph points on the coordinate plane to solve real-world and mathematical problems.	Write and interpret numerical expressions.
Perform operations with multi-digit whole numbers and with decimals to hundredths. ¹¹	Classify two-dimensional figures into categories based on their properties.	Analyze patterns and relationships.
Use equivalent fractions as a strategy to add and subtract fractions.		Represent and interpret data.
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.		
Convert like measurement units within a given measurement system.		
Geometric measurement:		

¹¹ Cluster contains a fluency standard.



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understand concepts of volume
and relate volume to
multiplication and to addition.



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Grade 6 Standards Analysis

Examples of Key Advances from Grade 5 to Grade 6

- Students' prior understanding of and skill with multiplication, division, and fractions contribute to their study of ratios, proportional relationships, and unit rates (6.RP).
- Students begin using properties of operations systematically to work with variables, variable expressions, and equations (6.EE).
- Students begin working with the system of rational numbers, including using positive and negative numbers to describe quantities (6.NS.5), extending the number line and coordinate plane to represent rational numbers and ordered pairs (6.NS.6), and understanding ordering and absolute value of rational numbers (6.NS.7).
- Having worked with measurement data in previous grades, students begin to develop notions of statistical variability, summarizing and describing distributions (6.SP).

Fluency Expectations and Examples of Culminating Standards

- 6.NS.2 Students fluently divide multidigit numbers using the standard algorithm. This is the culminating standard for several years' worth of work with division of whole numbers.
- 6.NS.3 Students fluently add, subtract, multiply, and divide multidigit decimals using the standard algorithm for each operation. This is the culminating standard for several years' worth of work relating to the domains of NBT, OA, and NF.
- 6.NS.1 Students interpret and compute quotients of fractions and solve word problems involving division of fractions by fractions. This completes the extension of operations to fractions.

Examples of Major Within-Grade Dependencies

- Equations of the form $px = q$ (6.EE.7) are unknown-factor problems; the solution is the quotient of a fraction by a fraction (6.NS.1).
- Solving problems by writing and solving equations (6.EE.7) involves not only an appreciation of how variables are used (6.EE.6) and what it means to solve an

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equation (6.EE.5), but also some ability to write, read, and evaluate expressions in which letters stand for numbers (6.EE.2).

- Students must be able to place rational numbers on a number line (6.NS.7) before they can place ordered pairs of rational numbers on a coordinate plane (6.NS.8). The former standard about ordering rational numbers is much more fundamental.

Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- Students' work with ratios and proportional relationships (6.RP) can be combined with their work in representing quantitative relationships between dependent and independent variables (6.EE.9).
- Plotting rational numbers in the coordinate plane (6.NS.8) is part of analyzing proportional relationships (6.RP.3a, 7.RP.2) and will become important for studying linear equations (8.EE.8) and graphs of functions (8.F).¹²
- Students use their skill in recognizing common factors (6.NS.4) to rewrite expressions (6.EE.3).
- Writing, reading, evaluating, and transforming variable expressions (6.EE.1–4) and solving equations and inequalities (6.EE.7–8) can be combined with use of the volume formulas $V = lwh$ and $V = bh$ (6.G.2).
- Working with data sets can connect to estimation and mental computation. For example, in a situation where there are 20 different numbers that are all between 8 and 10, one might quickly estimate the sum of the numbers as $9 \times 20 = 180$.

Examples of Opportunities for In-Depth Focus

- 6.RP.3 When students work toward meeting this standard, they use a range of reasoning and representations to analyze proportional relationships.
- 6.NS.1 This is a culminating standard for extending multiplication and division to fractions.

¹² While not required by the standards, it might be considered valuable to expose students to time series data and to time graphs as an appealing way to work with rational numbers in the coordinate plane (6.NS.8). For example, students could create time graphs of temperature measured each hour over a 24-hour period in a place where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day.

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- 6.NS.8 When students work with rational numbers in the coordinate plane to solve problems, they combine and consolidate elements from the other standards in this cluster.
- 6.EE.3 By applying properties of operations to generate equivalent expressions, students use properties of operations that they are familiar with from previous grades' work with numbers — generalizing arithmetic in the process.
- 6.EE.7 When students write equations of the form $x + p = q$ and $px = q$ to solve real-world and mathematical problems, they draw on meanings of operations that they are familiar with from previous grades' work. They also begin to learn algebraic approaches to solving problems.¹³

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- Reading and transforming expressions involves seeing and making use of structure (MP.7).
- The sequence of steps in the solution of an equation is a logical argument that students can construct and critique (MP.3).
- Thinking about the point $(1, r)$ in a graph of a proportional relationship with unit rate r involves reasoning abstractly and quantitatively (MP.2).
- Area, surface area, and volume present modeling opportunities (MP.4).
- Students think with precision (MP.6) and reason quantitatively (MP.2) when they use variables to represent numbers and write expressions and equations to solve a problem (6.EE.6–7).
- Working with data gives students an opportunity to use appropriate tools strategically (MP.5). For example, spreadsheets can be powerful for working with a data set with dozens or hundreds of data points.

¹³ For example, suppose Daniel went to visit his grandmother, who gave him \$5.50. Then he bought a book costing \$9.20 and had \$2.30 left. To find how much money he had before visiting his grandmother, an algebraic approach leads to the equation $x + 5.50 - 9.20 = 2.30$. An arithmetic approach without using variables at all would be to begin with 2.30, then add 9.20, then subtract 5.50. This yields the desired answer, but students will eventually encounter problems in which arithmetic approaches are unrealistically difficult and algebraic approaches must be used.



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Instructional Emphases by Cluster

The five domains in grade 6 contain 10 clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
Understand ratio concepts and use ratio reasoning to solve problems.	Compute fluently with multi-digit numbers and find common factors and multiples. ¹⁴	Develop understanding of statistical variability.
Apply and extend previous understandings of multiplication and division to divide fractions by fractions.	Apply and extend previous understandings of numbers to the system of rational numbers.	Summarize and describe distributions.
Apply and extend previous understandings of arithmetic to algebraic expressions.	Solve real-world and mathematical problems involving area, surface area, and volume.	
Reason about and solve one-variable equations and inequalities.		
Represent and analyze quantitative relationships between dependent and independent variables.		

¹⁴ Cluster contains fluency standards.

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Grade 7 Standards Analysis

Examples of Key Advances from Grade 6 to Grade 7

- In grade 6, students learned about rational numbers and the kinds of quantities they can be used to represent; they also learned about absolute value and ordering of rational numbers, including in real-world contexts. In grade 7, students will add, subtract, multiply, and divide within the system of rational numbers.
- Students grow in their ability to analyze proportional relationships. They decide whether two quantities are in a proportional relationship (7.RP.2a); they work with percents (7.RP.3); they analyze proportional relationships and solve problems involving unit rates associated with ratios of fractions (e.g., if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, the unit rate is the complex fraction $\frac{1/2}{1/4}$ miles per hour or 2 miles per hour); and they analyze proportional relationships in geometric figures (7.G.1).
- Students solve a variety of problems involving angle measure, area, surface area, and volume (7.G.4–6).

Fluency Expectations and Examples of Culminating Standards

- 7.EE.3 Students solve multistep problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. This work is the culmination of many progressions of learning in arithmetic, problem solving, and mathematical practices.
- 7.EE.4 In solving word problems leading to one-variable equations of the form $px + q = r$ and $p(x + q) = r$, students solve the equations fluently. This will require fluency with rational number arithmetic (7.NS.1–3), as well as fluency to some extent with applying properties operations to rewrite linear expressions with rational coefficients (7.EE.1).
- 7.NS.1–2 Adding, subtracting, multiplying, and dividing rational numbers is the culmination of numerical work with the four basic operations. The number system will continue to develop in grade 8, expanding to become the real numbers by the introduction of irrational numbers, and will develop further in high school, expanding to become the complex numbers with the introduction of imaginary numbers. Because there are no specific standards for rational number arithmetic in later grades and because so much other work in grade 7 depends on rational number arithmetic (see below), fluency with rational number arithmetic should be the goal in grade 7.

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Examples of Major Within-Grade Dependencies

- Meeting standard 7.EE.3 in its entirety will involve using rational number arithmetic (7.NS.1–3) and percents (7.RP.3). Work leading to meeting this standard could be organized as a recurring activity that tracks the students’ ongoing acquisition of new skills in rational number arithmetic and percents.
- Because rational number arithmetic (7.NS.1–3) underlies the problem solving detailed in 7.EE.3 as well as the solution of linear expressions and equations (7.EE.1–2, 4), this work should likely begin at or near the start of the year.
- The work leading to meeting standards 7.EE.1–4 could be divided into two phases, one centered on addition and subtraction (e.g., solving $x + q = r$) in relation to rational number addition and subtraction (7.NS.1) and another centered on multiplication and division (e.g., solving $px + q = r$ and $p(x + q) = r$) in relation to rational number multiplication and division (7.NS.2).

Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- Students use proportional reasoning when they analyze scale drawings (7.G.1).
- Students use proportional reasoning and percentages when they extrapolate from random samples and use probability (7.SP.6, 8).

Examples of Opportunities for In-Depth Focus

- 7.RP.2 Students in grade 7 grow in their ability to recognize, represent, and analyze proportional relationships in various ways, including by using tables, graphs, and equations.
- 7.NS.3 When students work toward meeting this standard (which is closely connected to 7.NS.1 and 7.NS.2), they consolidate their skill and understanding of addition, subtraction, multiplication, and division of rational numbers.
- 7.EE.3 This is a major capstone standard for arithmetic and its applications.
- 7.EE.4 Work toward meeting this standard builds on the work that led to meeting 6.EE.7 and prepares students for the work that will lead to meeting 8.EE.7.

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- 7.G.6 Work toward meeting this standard draws together grades 3–6 work with geometric measurement.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- When students compare arithmetic and algebraic solutions to the same problem (7.EE.4a), they are identifying correspondences between different approaches (MP.1).
- Solving an equation such as $4 = 8(x - \frac{1}{2})$ requires students to see and make use of structure (MP.7), temporarily viewing $x - \frac{1}{2}$ as a single entity.
- When students notice when given geometric conditions determine a unique triangle, more than one triangle, or no triangle (7.G.2), they have an opportunity to construct viable arguments and critique the reasoning of others (MP.3). Such problems also present opportunities for using appropriate tools strategically (MP.5).
- Proportional relationships present opportunities for modeling (MP.4). For example, the number of people who live in an apartment building might be taken as proportional to the number of stories in the building for modeling purposes.

Instructional Emphases by Cluster

The five domains in grade 7 contain nine clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
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Analyze proportional relationships and use them to solve real-world and mathematical problems.

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Use properties of operations to generate equivalent expressions.

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.¹⁵

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Draw, construct and describe geometrical figures and describe the relationships between them.

Use random sampling to draw inferences about a population.

Investigate chance processes and develop, use, and evaluate probability models.

Draw informal comparative inferences about two populations.

¹⁵ Cluster contains a fluency standard.

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Grade 8 Standards Analysis

Examples of Key Advances from Grade 7 to Grade 8

- Students build on previous work with proportional relationships, unit rates, and graphing to connect these ideas and understand that the points (x,y) on a nonvertical line are the solutions of the equation $y = mx + b$, where m is the slope of the line as well as the unit rate of a proportional relationship (in the case $b = 0$). Students also formalize their previous work with linear relationships by working with functions — rules that assign to each input exactly one output.
- By working with equations such as $x^2 = 2$ and in geometric contexts such as the Pythagorean Theorem, students enlarge their concept of number beyond the system of rationals to include irrational numbers. They represent these numbers with radical expressions and approximate these numbers with rationals.

Fluency Expectations and Examples of Culminating Standards

- 8.EE.7** Students have been working informally with one-variable linear equations since as early as kindergarten. This important line of development culminates in grade 8 with the solution of general one-variable linear equations, including cases with infinitely many solutions or no solutions and cases requiring algebraic manipulation using properties of operations. Coefficients and constants in these equations may be any rational numbers.
- 8.G.9** When students learn to solve problems involving volumes of cones, cylinders, and spheres — together with their previous grade 7 work in angle measure, area, surface area, and volume (7.G.4–6) — they will have acquired a well-developed set of geometric measurement skills. These skills, along with proportional reasoning (7.RP) and multistep numerical problem solving (7.EE.3), can be combined and used in flexible ways as part of modeling during high school — not to mention after high school for college and careers.¹⁶

Examples of Major Within-Grade Dependencies

- An important development takes place in grade 8 when students make connections between proportional relationships, lines, and linear equations (8.EE, second cluster). Making these connections depends on prior grades' work, including 7.RP.2 and 6.EE.9.

¹⁶ See <http://commoncoretools.wordpress.com/2011/06/15/essay-by-jason-zimba-on-pinnacle-standards/>.

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There is also a major dependency within grade 8 itself: The angle-angle criterion for triangle similarity underlies the fact that a nonvertical line in the coordinate plane has equation $y = mx + b$.¹⁷ Therefore, students must do work with congruence and similarity (8.G.1–5) before they are able to justify the connections among proportional relationships, lines, and linear equations. Hence the indicated geometry work should likely begin at or near the very start of the year.¹⁸

- Much of the work of grade 8 involves lines, linear equations, and linear functions (8.EE.5–8; 8.F.3–4; 8.SP.2–3). Irrational numbers, radicals, the Pythagorean Theorem, and volume (8.NS.1–2; 8.EE.2; 8.G.6–9) are nonlinear in nature. Curriculum developers might choose to address linear and nonlinear bodies of content somewhat separately. An exception, however, might be that when addressing functions, pervasively treating linear functions as separate from nonlinear functions might obscure the concept of function *per se*. There should also be sufficient treatment of nonlinear functions to avoid giving students the misleading impression that all functional relationships are linear (see also 7.RP.2a).

Examples of Opportunities for Connections Among Standards, Clusters, or Domains

- Students' work with proportional relationships, lines, linear equations, and linear functions can be enhanced by working with scatter plots and linear models of association in bivariate measurement data (8.SP.1–3).
- Work with the number system in this grade (8.NS.1–2) is intimately related to work with radicals (8.EE.2), and both of these may be connected to the Pythagorean Theorem (8.G, second cluster) as well as to volume problems (8.G.9), e.g., in which a cube has known volume but unknown edge lengths.

Examples of Opportunities for In-Depth Focus

- 8.EE.5 When students work toward meeting this standard, they build on grades 6–7 work with proportions and position themselves for grade 8 work with functions and the equation of a line.
- 8.EE.7 This is a culminating standard for solving one-variable linear equations.

¹⁷ See p. 12 of the *Progression for Expressions and Equations*:

http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_ee_2011_04_25.pdf.

¹⁸ Note that the geometry cluster “Understand congruence and similarity using physical models, transparencies, or geometry software” supports high school work with congruent triangles and congruent figures.

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- 8.EE.8 When students work toward meeting this standard, they build on what they know about two-variable linear equations, and they enlarge the varieties of real-world and mathematical problems they can solve.
- 8.F.2 Work toward meeting this standard repositions previous work with tables and graphs in the new context of input/output rules.
- 8.G.7 The Pythagorean Theorem is useful in practical problems, relates to grade-level work in irrational numbers, and plays an important role mathematically in coordinate geometry in high school.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- When students convert a fraction such as $\frac{1}{7}$ to a decimal, they might notice that they are repeating the same calculations and conclude that the decimal repeats. Similarly, by repeatedly checking whether points are on a line through (1,2) with slope 3, students might abstract the equation of the line in the form $(y - 2)/(x - 1) = 3$. In both examples, students look for and express regularity in repeated reasoning (MP.8).
- The Pythagorean Theorem can provide opportunities for students to construct viable arguments and critique the reasoning of others (e.g., if a student in the class seems to be confusing the theorem with its converse) (MP.3).
- Solving an equation such as $3(x - \frac{1}{2}) = x + 2$ requires students to see and make use of structure (MP.7).
- Much of the mathematics in grade 8 lends itself to modeling (MP.4). For example, standard 8.F.4 involves modeling linear relationships with functions.
- Working with scientific notation (8.EE.4) presents opportunities for strategically using appropriate tools (MP.5). For example, a computation such as $(1.73 \times 10^{-4}) \cdot (1.73 \times 10^{-4})$ can be done quickly with a calculator by simply squaring 1.73 and then using properties of exponents to determine the exponent of the product by inspection.



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Instructional Emphases by Cluster

The five domains in grade 8 contain 10 clusters. Some clusters require greater emphasis than others. The following table shows the relative emphasis for each cluster. No material in the standards should be excluded.

First Priority	Second Priority	Third Priority
Work with radicals and integer exponents.	Know that there are numbers that are not rational, and approximate them by rational numbers.	Investigate patterns of association in bivariate data.
Understand the connections between proportional relationships, lines, and linear equations.	Use functions to model relationships between quantities.	
Analyze and solve linear equations and pairs of simultaneous linear equations. ¹⁹	Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.	
Define, evaluate, and compare functions.		
Understand and apply the Pythagorean Theorem.		
Understand congruence and similarity using physical models, transparencies, or geometry software.		

¹⁹ Cluster includes a standard with elements of fluency (solving simple 2x2 systems by inspection).



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Introduction to the High School Standards Analysis

The Standards for Mathematical Practice are common to both K–8 and high school, but the Standards for Mathematical Content are organized differently in K–8 than in high school. In grades K–8, the content standards are organized in a yearly sequence. In high school, the content standards consist of about three years’ worth of material organized not by year but rather by conceptual category (Functions, Algebra, etc.).

The Partnership for Assessment of Readiness for College and Careers (PARCC) Model Content Frameworks provide an analysis of the high school standards using terms similar to those used for the grades 3–8 standards analyses. This is done by organizing the high school standards into courses: Algebra I–Geometry–Algebra II and Mathematics I–Mathematics II–Mathematics III.

Previous drafts of the Model Content Frameworks for mathematics provided little or no detail about high school courses based on the standards. This public review draft adds more detail about courses, including suggesting priorities in the course introductions, the “Key Advances,” “Connections to Practices,” and “In-Depth Focus” sections for each course, but it does not specify full details of the courses. Further specification will be provided in the future to the degree necessary so that courses can be aligned properly with the assessments being designed by the consortium states. The Model Content Frameworks provide initial, high-level guidance.

There are two sections to the High School Standards Analysis:

1. **General analysis** of the high school standards: analysis that bears on all courses and/or is best presented independently of any particular organization of the standards into courses given the current stage of high school assessment design.
2. **Course-specific analysis** of the high school standards: analysis presented with a view toward the two high school course sequences.

As courses become more precisely defined, general analysis can be replaced by course-specific analysis in future versions of the frameworks.

General Analysis

Examples of Opportunities for Connections Among Standards, Clusters, Domains, or Conceptual Categories

- The standards identify a number of connections among conceptual categories.

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- *Connections among Algebra, Functions, and Modeling.* Expressions can define functions; determining an output value for a particular input involves evaluating an expression. Equivalent expressions define the same function. Asking when two different functions have the same value for the same input leads to an equation (e.g., for what x does $x^3 = 2x + 5$?); graphing the two functions allows for finding approximate solutions of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling. Because functions describe relationships among quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.
- *Connections between Geometry and Algebra.* The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation a tool for geometric understanding, modeling, and proof. Geometric transformations provide examples of how the notion of function can be used in geometric contexts; conversely, the effect of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific positive and negative values of k can be interpreted geometrically in terms of transformations on the graphs of the functions.
- *Connections among Statistics, Functions, and Modeling.* Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Examples of Opportunities for Connecting Mathematical Content and Mathematical Practices

- When students use algebra and functions to model a situation, the symbolic calculations they use and the conclusions they draw from those calculations are examples of decontextualizing and contextualizing (reasoning abstractly and quantitatively, MP.2). For example, students looking for a general method of comparing two rate plans with different rates and startup costs ($R_1 = ax + b$, $R_2 = cx + d$) might find the crossover point by working symbolically to solve the equation $ax + b = cx + d$, obtaining the formal solution $x_{\text{crossover}} = (d - b)/(a - c)$. Still thinking symbolically, students can notice that the expression for $x_{\text{crossover}}$ is undefined when $a = c$. Returning to the context, students can see that this makes sense: two rate plans with the same rate never cross; the better plan in this case is always the one with the lower startup cost. Returning again to the

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symbolic equation, students can see that in the case of equal rates ($a = c$), the equation for the crossover point reduces to $b = d$, an equation that is true for all x if and only if the two plans have the same startup cost ... in which case they are the same plan.

- When students transform expressions purposefully, they are looking for and making use of structure (MP.7).
- Often when modeling a situation, students can get started by working repetitively with numerical examples, and then they look for and express regularity in that repeated reasoning by writing equations or functions (MP.8).
- Throughout high school, students construct viable arguments and critique the reasoning of others (MP.3). As in geometry, important questions in advanced algebra cannot be answered definitively by checking evidence. Results about all objects of a certain type — the factor theorem for polynomials, for example — require general arguments. And deciding whether or not two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires arguments of a different sort.
- Capturing a situation with precise language (MP.6) can be a critical step toward modeling that situation mathematically. For example, when investigating loan payments, if students can articulate something like, “What you owe at the end of a month is what you owed at the start of the month, plus $\frac{1}{12}$ th of the yearly interest on that amount, minus the monthly payment,” they are well along a path that will let them construct a recursively defined function for calculating loan payments.
- There are many opportunities in high school to use appropriate tools strategically (MP.5). For example,
 - Students might use graphing calculators or software to gain understanding of the important fact that the graph of an equation in two variables often forms a curve (which could be a line) (A-REI.10). Students might also use graphing calculators and/or graphing software to gain understanding of the important technique of looking for solutions to equations of the form $f(x) = g(x)$ by graphing the solutions of the equations $y = f(x)$ and $y = g(x)$ in the coordinate plane and looking for intersections of the graphs (A-REI.11)
 - Students might use graphing calculators or software to experiment with cases of replacing a function $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific positive and negative values of k (F-BF.3).
 - Students might use spreadsheets or similar technology in modeling situations to compute and display recursively defined functions (e.g., a function that gives the



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balance B_n on a credit card after n months given the interest rate, starting balance, and regular monthly payment) (F-BF.1a; F-LE).

- Students might use a computer algebra system to transform or experiment with algebraic expressions (A-APR.6).
- When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data (Common Core State Standards, p. 72).
- Technology is usually necessary to work effectively with large data sets or with simulations having many iterations.

Instructional Emphases by Cluster

The four to six domains in each conceptual category contain anywhere from nine to 11 clusters (but 15 in the case of geometry). Some clusters require greater emphasis than others. Note that the Common Core State Standards Working Group was charged with basing the standards on evidence, and the actual demands of college and careers elevate some material in the standards to high importance, while making other material less important. (Nevertheless, students who intend to pursue STEM careers or take Advanced Placement or International Baccalaureate courses during high school should have a fairly firm command even of material in the second and third levels of priority.)

Beginning on the next page, a series of tables show the relative emphasis of each cluster for the high school standards as a whole. Because some clusters in the following tables include (+) standards, it should be noted that no (+) standards will contribute to a summative assessment score in the PARCC assessment system.

Note also that course-specific priorities will be provided when full details are available for high school courses.

Prioritization does not imply neglect of material. No material in the standards should be excluded. But clear priorities can help ensure that curricula attend properly to the relative importance of the content. Note that the prioritization is in terms of cluster headings. Setting priorities at the cluster level is a way to talk with reasonable specificity about mathematical content while still maintaining the coherence of the mathematics.



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Number and Quantity

First Priority	Second Priority	Third Priority
Reason quantitatively and use units to solve problems.	Perform arithmetic operations with complex numbers.	Represent complex numbers and their operations on the complex plane.
Extend the properties of exponents to rational exponents.	Use properties of rational and irrational numbers.	Use complex numbers in polynomial identities and equations.
		Represent and model with vector quantities.
		Perform operations on vectors.
		Perform operations on matrices and use matrices in applications.

Algebra

First Priority	Second Priority	Third Priority
Interpret the structure of expressions.	Rewrite rational expressions.	Use polynomial identities to solve problems.
Write expressions in equivalent forms to solve problems.	Represent and solve equations and inequalities graphically.	
Perform arithmetic operations on polynomials.		
Understand the relationship between zeros and factors of polynomials.		
Create equations that describe numbers or relationships.		
Understand solving equations as a process of reasoning and explain the reasoning.		
Solve equations and inequalities in one variable.		
Solve systems of equations.		



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Functions

First Priority	Second Priority	Third Priority
Understand the concept of a function and understand function notation.	Build new functions from existing functions.	Extend the domain of trigonometric functions using the unit circle.
Interpret functions that arise in applications in terms of the context.		Model periodic phenomena with trigonometric functions.
Analyze functions using different representations.		Prove and apply trigonometric identities.
Build a function that models a relationship between two quantities.		
Construct and compare linear, quadratic, and exponential models and solve problems.		
Interpret expressions for functions in terms of the situation they model.		

Geometry

First Priority	Second Priority	Third Priority
Prove geometric theorems.	Experiment with transformations in the plane.	Prove theorems involving similarity.
Use coordinates to prove simple theorems algebraically.	Understand congruence in terms of rigid motions.	Apply trigonometry to general triangles.
Define trigonometric ratios and solve problems involving right triangles.	Make geometric constructions.	Explain volume formulas and use them to solve problems.
Apply geometric concepts in modeling situations.	Understand and apply theorems about circles.	Visualize relationships between two-dimensional and three-dimensional objects.
	Find arc lengths and areas of sectors of circles.	Translate between the geometric description and the equation for a conic section. (Here because of circles.)
	Understand similarity in terms of similarity transformations.	



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Statistics and Probability

First Priority	Second Priority	Third Priority
Summarize, represent, and interpret data on a single count or measurement variable.	Understand and evaluate random processes underlying statistical experiments.	Understand independence and conditional probabilities of compound events in a uniform probability model.
Summarize, represent, and interpret data on two categorical and quantitative variables.	Interpret linear models.	Use the rules of probability to compute probabilities of compound events in a uniform probability model.
Make inferences and justify conclusions from sample surveys, experiments, and observational studies.		Calculate expected values and use them to solve problems.
		Use probability to evaluate outcomes of decisions.



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Course-Specific Analyses

Each course is introduced with a high-level narrative. This narrative gives a sense of the overall course goals. The description is not intended to be exhaustive.

Course-specific analysis is then provided in the following categories:

Examples of Key Advances from Previous Grades or Courses

- Highlights some of the major steps in the progression of increasing knowledge and skill from year to year. Note that each key advance in mathematical content also corresponds to a widening scope of problems that students can solve. Examples of key advances are highlighted to stress the need for curricula to treat topics in ways that take into account where students have been in previous grades or courses and where they will be going in subsequent courses.¹

Fluency Recommendations

- The high school standards do not set explicit expectations for fluency, but fluency is important in high school mathematics. For example, fluency in algebra can help students get past the need to manage computational details, so they can observe structure and patterns in problems. Such fluency can also allow for smooth progress beyond the college and career readiness threshold toward STEM readiness. Therefore, this section makes recommendations about fluencies that can serve students well as they learn and apply mathematics. These fluencies are highlighted to stress the need for curricula to provide sufficient supports and opportunities for practice to help students gain fluency. Fluency is not meant to come at the expense of understanding; it is an outcome of a progression of learning and thoughtful practice. Curricula must provide the conceptual building blocks that develop in tandem with skill along the way to fluency.

Discussion of Mathematical Practices in Relation to Course Content

- Highlights some of the mathematical practices and describes how they play a role in each course.² These examples are provided to stress the need to connect content and practices, as required by the standards.

¹ See the *Progressions* documents for additional information about progressions in the standards.

² See the *Progressions* documents for additional examples.

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- In addition to the examples provided in each course, the following are some general comments about connecting content and practices:
 - Connecting content and practices happens in the context of ***working on problems***. The very first standard for mathematical practice is to make sense of problems and persevere in solving them (**MP.1**).
 - The Standards for Mathematical Practices interact and overlap with each other. ***They are not a checklist.***

Examples of Major Within-Course Dependencies (*more to come when courses are fully defined*)

- Highlights cases in which a body of content within a given course depends conceptually or logically on another body of content within that same course. Examples of within-course dependencies are highlighted to stress the need for curricula to organize material coherently within each given course. (Because of space limitations, only examples of large-scale dependencies are described here; but coherence is important for dependencies that exist at finer grain size as well.)

Examples of Opportunities for In-Depth Focus

- Highlights some individual standards that play an important role in the high school content. Curriculum developers may choose to give the indicated mathematics an especially in-depth treatment, as measured for example by the number of days; the quality of classroom activities for exploration and reasoning; the amount of student practice; and the rigor of expectations for depth of understanding or mastery of skills.

Please Note

- The words *examples* and *opportunities* in the above categories emphasize that the analysis provided in each category is not exhaustive. For example, there are many opportunities to connect mathematical content and practices in every course; there are many opportunities for in-depth focus in every grade; and so on. A comprehensive description of these features of the standards would be hundreds of pages long. ***The analyses given here should be thought of as starting points.***
- Always refer back to the *Common Core State Standards for Mathematics* for exact language about student expectations.

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Algebra I Standards Analysis

Students in Algebra I fully master linear equations and linear functions, especially the algebra-geometry interplay regarding slope and graphs. Students also work intensively to master quadratic functions, both from an algebraic and formal perspective as well as in the context of modeling. The work that students do with quadratic functions is connected with and reinforces their work in quadratic equations, polynomial arithmetic, and seeing structure in expressions. From an applications perspective, quadratic functions provide opportunities for the study of optimization, which is an important aspect of modeling. Working intensively with linear and quadratic expressions, equations, and functions in Algebra I enables students to focus and master this material.

At the same time, however, students in Algebra I encounter general principles and techniques that apply much more generally than in the linear or quadratic case — for example, learning that the graph of an equation in two variables often forms a curve, which could be a line (A-REI.10). Thus, although most of Algebra I focuses on linear and quadratic equations and functions, the course does include concepts that apply more generally and therefore need to be illustrated beyond the linear and quadratic case.

Examples of Key Advances from Grades K–8

- Having already extended arithmetic from whole numbers to fractions (grades 4–6) and from fractions to rational numbers (grade 7), students in grade 8 encountered particular irrational numbers such as $\sqrt{5}$ or π^2 . In Algebra I, students will develop an understanding of the real number *system*. For more on the extension of number systems, see “Numbers and Number Systems” on page 58 of the standards.
- Students in grade 8 worked with integer exponents. In Algebra I, students will extend the properties of exponents to positive real numbers raised to rational powers (N-RN.1,2).
- Students in middle grades worked with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I, students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight (N-Q).
- Themes beginning in middle school algebra continue and deepen during high school. As early as grades 6 and 7, students began to use the properties of operations to generate equivalent expressions (6.EE.3, 7.EE.1). By grade 7, they began to recognize that rewriting expressions in different forms could be useful in problem solving.

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(7.EE.2). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called “mindful manipulation.”³

- Students in grade 8 extended their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students will master linear and quadratic functions. Students encounter other kinds of functions, so as to ensure that general principles are perceived in generality, as well as to enrich the range of quantitative relationships considered in problems.
- Students in grade 8 connected their knowledge about proportional relationships, lines, and linear equations (8.EE.5,6). In Algebra I, students solidify their understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane,
 - The graph of any linear equation in two variables is a line.
 - Any line is the graph of a linear equation in two variables.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills they first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.

Discussion of Mathematical Practices in Relation to Course Content

Two overarching practices relevant to Algebra I are:

- **Make sense of problems and persevere in solving them (MP.1).**
- **Model with mathematics (MP.4).**

Indeed, other mathematical practices in Algebra I might be seen as contributing specific elements of these two.

- **Reason abstractly and quantitatively (MP.2).** This practice standard refers to one of the hallmarks of algebraic reasoning, the process of decontextualization and contextualization. Much of elementary algebra involves creating abstract algebraic

³ See, e.g., “Mindful Manipulation,” in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009).

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models of situations (A-CED, F-BF) and then transforming the models via algebraic calculations (A-SSE, A-APR, F-IF) to reveal properties of the situations.

- **Attend to precision (MP.6).** In algebra, the habit of using precise language not only is a mechanism for effective communication it is also a tool for understanding and solving problems. Describing an idea precisely (A-CED, A-REI) helps you understand the idea in new ways.
- **Look for and make use of structure (MP.7).** For example, writing $49x^2 + 35x + 6$ as $(7x)^2 + 5(7x) + 6$, a practice many teachers refer to as “chunking,” highlights the structural similarity between this expression and $z^2 + 5z + 6$, leading to a factorization of the original: $((7x) + 3)((7x) + 2)$ (A-SSE, A-APR).
- **Look for and express regularity in repeated reasoning (MP.8).** Creating equations or functions to model situations is harder for many students than working with the resulting expressions. An effective way to help students develop the skill of describing general relationships is to work through several specific examples and then express what they are doing with algebraic symbolism (A-CED).
- **Use appropriate tools strategically (MP.5).** Spreadsheets, a function modeling language, graphing tools, and many other technologies can be used strategically to gain understanding of the ideas expressed by individual content standards and to model with mathematics.

Fluency Recommendations

- | | |
|----------|---|
| A/G | Algebra I students become fluent in solving characteristic problems involving the analytic geometry of lines, such as writing down the equation of a line given a point and a slope. Such fluency can support them in solving less routine mathematical problems involving linearity, as well as in modeling linear phenomena (including modeling using systems of linear inequalities in two variables). |
| A-APR.1 | Fluency in adding, subtracting, and multiplying polynomials supports students throughout their work in algebra, as well as in their symbolic work with functions. Manipulation can be more mindful when it is fluent. |
| A-SSE.1b | Fluency in transforming expressions and chunking (seeing parts of an expression as a single object) is essential in factoring, completing the square, and other mindful algebraic calculations. |

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Examples of Major Within-Course Dependencies

- A solid understanding of the correspondence between an equation in two variables and its Cartesian graph is the underpinning for the techniques for graphing lines and quadratics, and it helps students understand what is meant by the “graph of a function.”
- Creating equations and building functions helps students interpret these same objects.

Examples of Opportunities for In-Depth Focus

- N-NQ.1** Quantitative problems involving reasoning with units are common in applications.
- A-SSE.2–3** Identifying ways to rewrite an expression based on its structure, and doing so intentionally in the context of solving a problem, are habits of mind that can support students in high school mathematics up to and beyond the college and career readiness threshold.
- A-APR.1** Adding, subtracting, and multiplying polynomials is a key skill in high school students’ work with algebra and functions.
- A-CED.1–2** Creating equations and inequalities to solve problems is a core modeling capacity.
- A-REI.4** Solving quadratic equations in one variable incorporates a range of solution methods that depend on facility with both manipulating expressions and reasoning about equations.
- F-IF.8** Rewriting quadratic function expressions to reveal properties of a function is a connection point between the conceptual categories of Algebra and Functions.
- F-IF.9** Being able to compare properties of two functions each represented in a different way is a sign that students are grasping mappings as mathematical objects.

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Geometry Standards Analysis

High school Geometry corresponds closely to the Geometry conceptual category in the high school standards. Thus, the course involves working with congruence (G-CO), similarity (G-SRT), right triangle trigonometry (in G-SRG), geometry of circles (G-C), analytic geometry in the coordinate plane (G-GPE), and geometric measurement (G-GMD) and modeling (G-MG).

Examples of Key Advances from Previous Grades or Courses

- Whereas concepts such as rotation, reflection, and translation were treated in the grade 8 standards mostly in the context of hands-on activities, and with an emphasis on geometric intuition, high school Geometry will put equal weight on precise definitions.
- In grades K–8, students worked with a variety of geometric measures (length, area, volume, angle, surface area, and circumference). In high school Geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (MP.4).
- The skills that students develop in Algebra I around simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use of the Pythagorean Theorem.
- In grade 8, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In high school Geometry, students will build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (G-GPE.1).

Discussion of Mathematical Practices in Relation to Course Content

- **Construct viable arguments and critique the reasoning of others (MP.3).** While all of high school mathematics should work to help students see the importance and usefulness of deductive arguments, geometry is an ideal arena for developing the skill of creating and presenting proofs (G-CO.9.10). One reason is that conjectures about geometric phenomena are often about infinitely many cases at once — for example, *every* angle inscribed in a semicircle is a right angle — so that such results cannot be established by checking every case (G-C.2).
- **Attend to precision (MP.6).** Teachers might use the activity of creating definitions as a way to help students see the value of precision. While this is possible in every course, the activity has a particularly visual appeal in geometry. For example, a class can build

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the definition of *quadrilateral* by starting with a rough idea (“four sides”), gradually refining the idea so that it rules out figures that do not fit the intuitive idea. Another place in geometry where precision is necessary and useful is in the refinement of conjectures so that initial conjectures that are not correct can be salvaged — two angle measures and a side length do not determine a triangle, but a certain configuration of these parts leads to the angle-side-angle theorem (G-CO.8).

- **Look for and make use of structure (MP.7).** Seeing structure in geometric configurations can lead to insights and proofs. This often involves the creation of auxiliary lines not originally part of a given figure. Two classic examples are the construction of a line through a vertex of a triangle parallel to the opposite side as a way to see that the angle measures of a triangle add to 180 degrees, and the introduction of a symmetry line in an isosceles triangle to see that the base angles are congruent (G-CO.9.10). Another kind of hidden structure makes use of area as a device to establish results about proportions, such as the important theorem (and its converse) that a line parallel to one side of a triangle divides the other two sides proportionally (G-SRT.4).
- **Reason abstractly and quantitatively (MP.2).** Abstraction is used in geometry when, for example, students use a diagram of a specific isosceles triangle as an aid to reason about *all* isosceles triangles (G-CO.9). Quantitative reasoning in geometry involves the real numbers in an essential way: irrational numbers show up in work with the Pythagorean Theorem (G-SRT.8), area formulas often depend (subtly and informally) on passing to the limit, and real numbers are an essential part of the definition of dilation (G-SRT.1). The proper use of units can help students understand the effect of dilation on area and perimeter (N-Q.1).
- **Use appropriate tools strategically (MP.5).** Dynamic geometry environments can help students look for invariants in a whole class of geometric constructions, and the constructions in such environments can sometimes lead to an idea behind a proof of a conjecture.

Fluency Recommendations

- G-SRT.5 Fluency with the triangle congruence and similarity criteria will help students throughout their investigations of triangles, quadrilaterals, circles, parallelism, and trigonometric ratios. These criteria are necessary tools in many geometric modeling tasks.

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G-GPE.4.5.7. Fluency with the use of coordinates to establish geometric results, calculate length and angle, and use geometric representations as a modeling tool are some of the most valuable tools in mathematics and related fields.

G-co.12. Fluency with the use of construction tools, physical and computational, helps students draft a model of a geometric phenomenon and can lead to conjectures and proofs.

Examples of Major Within-Course Dependencies

- The progression from congruence to area to similarity can be used to put each of these topics on a logical footing: The basic assumptions that congruent figures have the same area and that area is invariant under finite dissection bring coherence to the formulas for calculating areas of polygonal regions. These formulas, along with results such as the fact that triangles with equal bases and height have the same area, can be used to prove properties of dilations and similarity.
- The triangle similarity criteria are necessary to develop the trigonometry of right triangles.

Examples of Opportunities for In-Depth Focus

G-co.1,9–11 These standards form a core of deductive reasoning in geometry.

G-GPE.4 When students prove simple theorems algebraically using coordinates, they construct arguments (**MP.3**) and integrate knowledge from multiple domains.

G-GPE.7 Being able to determine geometric measures in a coordinate setting is a useful skill in applications and plays a role in calculus.

G-C.1.2.3 The suite of theorems about circles, angles, arcs, and related proportions provides students with an example of a coherent mathematical theory (**MP.1**).

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Algebra II Standards Analysis

Students in Algebra II expand their repertoire by working with rational and exponential expressions; polynomial, exponential, and logarithmic functions; trigonometric functions with real number domain; and sequences and series. Exponential functions, trigonometric functions, and sequences and series all provide opportunities for modeling. As students encounter more and more varied mathematical expressions and functions, general principles they encountered in Algebra I remain relevant, unifying the material in the course.

Examples of Key Advances from Previous Grades or Courses

- In Algebra I, students added, subtracted, and multiplied polynomials. In Algebra II, students divide polynomials with remainder, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle school algebra continue and deepen during high school. As early as grade 6, students began thinking about solving equations as a process of reasoning (6.EE.5). This perspective continues throughout Algebra I and Algebra II (A-REI).⁴ “Reasoned solving” plays a role in Algebra II because the equations students encounter can have extraneous solutions (A-REI.2).
- In Algebra I, students met quadratic equations with no real roots. In Algebra II, they extend the real numbers to complex numbers, and one effect of this is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicities) two roots in the complex numbers.
- In grade 8, students learned the Pythagorean Theorem and used it to determine distances in a coordinate system (8.G.6–8). In Geometry, students proved theorems using coordinates (G-GPE.4–7). In Algebra II, students will build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (e.g., G-GPE.1).
- In Geometry, students began trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., S-ID.6). In a modeling context, they

⁴ See, e.g., “Reasoned Solving,” in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009).

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might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes.

Discussion of Mathematical Practices in Relation to Course Content

While all of the mathematical practice standards are important in all three courses, four are especially important in the Algebra II course:

- **Construct viable arguments and critique the reasoning of others (MP.3).** As in geometry, there are central questions in advanced algebra that cannot be answered definitively by checking evidence. There are important results about *all* functions of a certain type — the factor theorem for polynomial functions, for example — and these require general arguments (A-APR.2). Deciding whether or not two functions are equal on an infinite set cannot be settled by looking at tables or graphs; it requires arguments of a different sort (F-IF.8).
- **Attend to precision (MP.6).** As in the previous two courses, the habit of using precise language is not only a tool for effective communication — it is means for coming to understanding. For example, when investigating loan payments, if students can articulate something like, “What you owe at the end of a month is what you owed at the start of the month, plus $\frac{1}{12}$ th of the yearly interest on that amount, minus the monthly payment,” they are well along a path that will let them construct a recursively defined function for calculating loan payments (A-SSE.4).
- **Look for and make use of structure (MP.7).** The structure theme in Algebra I centered on seeing and using the structure of algebraic expressions. This continues in Algebra II, where students delve deeper into transforming expressions in ways that reveal meaning. The example given in the standards — that $x^4 - y^4$ can be seen as the difference of squares — is typical of this practice. This habit of seeing subexpressions as single entities will serve students well in areas such as trigonometry, where, for example, the factorization of $x^4 - y^4$ described above can be used to show that the functions $\cos^4 x - \sin^4 x$ and $\cos^2 x - \sin^2 x$ are, in fact, equal (A-SSE.2).

In addition, the standards call for attention to the structural similarities between polynomials and integers (A-APR.1). The study of these similarities can be deepened in Algebra II: Like integers, polynomials have a division algorithm, and division of polynomials can be used to understand the factor theorem, to transform rational expressions, to help solve equations, and to factor polynomials.

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- Look for and express regularity in repeated reasoning (MP.8).** Algebra II is the place where students can do a more complete analysis of sequences (F-IF.3), especially arithmetic and geometric sequences, and their associated series. Developing recursive formulas for sequences is facilitated by the practice of abstracting regularity for how you get from one term to the next and then giving a precise description of this process in algebraic symbols (F-BF.2). Technology can be a useful tool here: Most CAS systems allow one to model recursive function definitions in notation that is close to standard mathematical notation. And spreadsheets make natural the process of taking successive differences and running totals (MP.5).

The same thinking — finding and articulating the rhythm in calculations — can help students analyze mortgage payments, and the ability for getting a closed form for a geometric series lets them make a complete analysis of this topic. This practice is also a tool for using difference tables to find simple functions that agree with a set of data.

Algebra II is a course in which students can learn some technical methods for performing algebraic calculations and transformations, but sense-making is still paramount (MP.1). For example, analyzing Heron’s formula from geometry lets one connect the zeros of the expression to the degenerate triangles. As in Algebra I, the modeling practice is ubiquitous in Algebra II, enhanced by the inclusion of exponential and logarithmic functions as modeling tools (MP.4). Computer algebra systems provide students with a tool for modeling mathematical phenomena, experimenting with algebraic objects (e.g., sequences of polynomials), and reducing the computational overhead needed to investigate many classical and useful areas of algebra (MP.5).

Fluency Recommendations

- A-APR.6** This standard sets an expectation that students will divide polynomials with remainder by inspection in simple cases. For example, one can view the rational expression $\frac{x+4}{x+3}$ as

$$\frac{x+4}{x+3} = \frac{(x+3)+1}{x+3} = 1 + \frac{1}{x+3}.$$

- A-SSE.2** The ability to see structure in expressions and to use this structure to rewrite expressions is a key skill in everything from advanced factoring (e.g., grouping) to summing series to the rewriting of rational expressions in order to examine the end behavior of the corresponding rational function.

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- F-IF.3** Fluency in translating between recursive definitions and closed forms is helpful when dealing with many problems involving sequences and series, with applications ranging from fitting functions to tables, to problems in finance.

Examples of Major Within-Course Dependencies

- The “Euclidean property” of polynomials — that for polynomials $f(x)$ and $g(x)$, there are unique polynomials $q(x)$ and $r(x)$ so that the degree of $r(x)$ is less than that of $g(x)$ and so that $f(x) = g(x)q(x) + r(x)$ — is needed to prove the remainder theorem, and this in turn is necessary for the factor theorem. Both of these latter results are needed to analyze methods for fitting polynomials to finite sets of data.
- The factor theorem is a foundation for understanding the connection between factorization of polynomials and the solutions to the corresponding polynomial functions, as well as to the connection between the solutions to a polynomial equation and the coefficients of the polynomial.

Examples of Opportunities for In-Depth Focus

- A-SSE.2–3** Identifying ways to rewrite an expression based on its structure, and doing so intentionally in the context of solving a problem, is a habit of mind that can support students in high school mathematics up to and beyond the college and career readiness threshold.
- A-APR.2** The ability to apply the factor and remainder theorems allows students to rewrite expressions to reveal quantities of interest and to evaluate functions at a given input.
- A-APR.6–7** Dividing polynomials with remainder and working with rational expressions are central to advanced algebra.
- A-CED.1–2** Creating equations to solve problems is a core modeling capacity.
- F-LE.1** The distinction between linear phenomena and exponential phenomena is often relevant in modeling; the mathematical properties of linear and exponential functions are fundamental to calculus as well.
- A-SSE.2** Chunking (seeing parts of expressions as single objects) is essential in advanced factoring methods and in establishing trigonometric identities.



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- A-REI.1** With the extended repertoire of functions, which includes exponential, logarithmic, and trigonometric functions, there is a need for reasoned solving of equations that involve these functions.

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Mathematics I Standards Analysis

Students formalize and deepen their knowledge of linear equations and inequalities, creating expressions, equations and inequalities to represent linear relationships and constraints, using these to solve problems using graphical, numeric, and algebraic methods, and explaining the reasoning behind their solutions. This knowledge is then extended to exponential expressions and equations, moving from a constant rate of growth to a proportional rate of growth. Students create expressions and equations to represent exponential relationships and use them to solve problems approximately and, in simple cases, algebraically. They use properties of exponents to create equivalent forms of an expression that reveal properties of the quantity the expression represents.

Students also formalize and deepen their knowledge of linear functions and begin working with exponential functions. Modeling plays a central role in students' development of knowledge with both linear and exponential functions; exploring a range of contexts that can be modeled with linear and exponential relationships will provide both motivation and meaning to their work. Students understand key features of both classes of functions, including their respective rates of growth, and can represent them in a variety of ways. Students interpret key features of these functions in terms of a context and use this understanding to write function rules for exponential and linear functions. Students also relate linear and exponential models to arithmetic and geometric progressions, including writing them recursively—a skill that comes into play when writing functions to model a situation.

Students formalize their work with congruence transformations, defining transformations in terms of geometric objects and measurements and relating them to congruence. They understand the effects of composing transformations and use transformations to explore symmetries of geometric objects. They use tools and methods based on transformations to construct geometric objects, such as paper folding and use of dynamic geometry environments.

Within the conceptual area of Statistics and Probability, students represent data on a single count or measurement variable in a variety of ways in order to better understand measures of center and of spread. They explore associations between two measurement variables using a scatter plot, and they begin to use linear models to better understand those associations when appropriate. Throughout their work in Mathematics I, students define quantities and use units appropriately as a way to guide their solutions, and as well as using and interpreting scales in graphs and data displays, attending to appropriate levels of accuracy.

Examples of Key Advances from Grades K–8

- Students build on previous work with solving linear equations and systems of linear equations in two ways: (a) they extend to more formal solution methods, including attending to the structure of linear expressions, and (b) they solve linear

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- inequalities.
- Students extend their work with integral exponents to understanding rational exponents. This work also leads to simplifying exponential expressions and solving simple exponential equations.
 - Students formalize their understanding of the definition of a function begun in the middle grades, as well as their understanding of linear functions in particular, emphasizing understanding the structure of linear expressions. Students also begin to work exponential functions, comparing them to linear functions.
 - Work with congruence and similarity motions in grades 6–8 is formalized in students’ definition of the transformations and their properties. Students also consider sufficient conditions for congruence.
 - Work with the bivariate data and scatter plots in grades 6–8 is extended to working with lines of best fit.

Discussion of Mathematical Practices in Relation to Course Content

- **Modeling with mathematics (MP.4)** should be a particular focus as students see the purpose and meaning for working with linear and exponential equations and functions.
- **Using appropriate tools strategically (MP.5)** is also important as students explore those models in a variety of ways, including with technology. For example, students might be given a set of data points and experiment with graphing a line that fits the data.
- As Course 1 continues to develop a foundation for more formal reasoning, students should engage in the practice of **constructing viable arguments and critiquing the reasoning of others (MP.3)**.

Fluency Recommendations

- | | |
|-----|---|
| A/G | High school students should become fluent in solving characteristic problems involving the analytic geometry of lines, such as finding the equation of a line given a point and a slope. This fluency can support students in solving less routine mathematical problems involving linearity, as well as in modeling linear phenomena (including modeling using systems of linear inequalities in two variables). |
| G | High school students should become fluent in using geometric transformation to represent the relationships among geometric objects. They provide a powerful tool for visualizing relationships, as well as a foundation for exploring ideas both within geometry (e.g., symmetry) and outside of geometry (e.g., transformations of graphs). |

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- s Students should be able to create a visual representation of a data set that is useful in understanding possible relationships among variables.

Examples of Major Within-Course Dependencies

- Students encounter linear models as a part of their work in the domains of both Algebra and Functions. It is important that connections be made between these two domains.
- A similar connection can be made between the work with exponential models in the domains of Algebra and Functions. Work with exponential models should build on work with linear models.
- Study of linear associations in statistics and probability (s-ID.6c,7) builds on students' understanding of linear relationships (cf. F-LE.1).

Examples of Opportunities for In-Depth Focus

- N-NQ.1 Quantitative problems involving reasoning with units are common in applications.
- A-CED.1,2 Creating equations in two or more variables to represent relationships among quantities is a core modeling capacity.
- A-REI.3,11 Developing a deep understanding of what an equation is and what it means to solve an equation, both approximately and as a process of reasoning, are important in supporting students' use of algebra to solve a range of problems, including problems presented in a context.
- F-IF.9 Being able to compare properties of two functions each represented in a different way is a sign that students are grasping mappings as mathematical objects.
- F-LE.1 The distinction between linear phenomena and exponential phenomena is often relevant in modeling.
- G-CO.6-8 Taking a deeper look at congruence motions and how they relate to properties of figures, as well as criteria for congruence, is an important building block for students to move to more formal arguments in geometry.
- s-ID.1-3, 6 These techniques are prevalent in the sciences, social sciences, and technical careers.

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Mathematics II Standards Analysis

Students extend their work in algebra with a major emphasis on quadratic relationships. They create expressions and equations to represent situations involving quadratic relationships. They recognize the key features of a quadratic expression and rewrite a quadratic expression in various forms to reveal information about a situation involving a quadratic relationship. They solve quadratic equations using a variety of methods, including using a table or graph to approximate solutions and using algebraic techniques such as factoring and completing the square to find exact solutions. They derive and use the quadratic formula. Students graphically explore solving a system of a linear equation and a quadratic equation, and extend their algebraic techniques to the solution.

Students also extend their work in functions to quadratic and other functions. Students write a function rule representing a quadratic relationship and can also represent that relationship using a table of values or a graph. They use technology to explore transformations of a quadratic function, including horizontal and vertical translations and stretches, looking at the correspondence between changes to the formula and its effects on the graph. They understand important features of a quadratic function, transform functions rules into forms that reveal those features, and interpret those features in terms of a context. They explore the growth rate of a quadratic function and compare it to linear and exponential functions.

In statistics and probability, students consider using a broader range of functions to model a relationship between two quantitative variables and assess the fit of the model. They also begin an exploration of probability, including understanding independence and conditional probability and using the rules of probability to compute probabilities of compound events.

Students extend their work with congruence to similarity. They develop criteria for congruence and similarity based on their work with transformations, which begins a transition towards formal proof and argumentation in geometry. As a part of this transition, they develop formal definitions for basic geometric objects and begin to produce formal arguments about geometric relationships and use such arguments solve problems. They also see the value of representing contexts using coordinates and use coordinates to prove or establish results. Students' proofs focus on explaining their reasoning rather than following a particular form.

Examples of Key Advances from Mathematics I

- Students extend their previous work with linear and exponential expressions, equations, systems of equations, and inequalities to quadratic relationships.

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- A parallel extension occurs from linear and exponential functions to quadratic functions, where students also begin to analyze functions in terms of transformations.
- Students extend their work with numbers to the complex numbers, which are developed as solutions to quadratic equations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships.

Discussion of Mathematical Practices in Relation to Course Content

- **Modeling with mathematics (MP.4)** should be a particular focus as students see the purpose and meaning for working with quadratic equations and functions, including **using appropriate tools strategically (MP.5)**.
- As students explore a variety of ways to represent quadratic expressions, they should **look for and make use of structure (MP.7)**.
- As their ability to create and use formal mathematical arguments grows, increased emphasis is placed on students' ability to attend to precision (**MP.6**), as well as to **construct viable arguments and critique the reasoning of others (MP.3)**.

Fluency Recommendations

- F/S** Fluency in graphing functions (including linear, quadratic, and exponential) and interpreting key features of the graphs in terms of their function rules and a table of value, as well as recognizing a relationship (including a relationship within a data set), fits one of those classes. This forms a critical base for seeing the value and purpose of mathematics, as well as for further study in mathematics.
- A-APR.1** Fluency in adding, subtracting, and multiplying polynomials supports students throughout their work in algebra, as well as in their symbolic work with functions. Manipulation can be more mindful when it is fluent.
- G-SRT.5** Fluency with the triangle congruence and similarity criteria will help students throughout their investigations of triangles, quadrilaterals, circles, parallelism, and trigonometric ratios. These criteria are necessary tools in geometric modeling.

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Examples of Major Within-Course Dependencies

- Deriving the quadratic formula (A-REI.4a) depends on being able to rewrite a quadratic expression by completing the square (A-SSE.1,3a, 3b).
- Introducing complex numbers as solutions to quadratic equations (N-CN.7; A-REI.4b(iii)) requires use of algebraic methods to solve quadratic equations (A-REI.4b).
- Understanding trigonometric ratios as properties of similar triangles (G-SRT.6) requires being able to prove problems related to similar triangles (G-SRT.5).
- Exploration of quadratic relationships in data on two measurement variables (S-ID.6) depends on understanding key features of a quadratic function and being able to interpret them in terms of a context (F-IF.4).

Examples of Opportunities for In-Depth Focus

- A-SSE.3 Identifying ways to rewrite a quadratic expression based on its structure to reveal key aspects of the situation being modeled, as well as to solve a quadratic equation, is an important habit of mind that can support students in high school mathematics up to and beyond the college and career readiness threshold.
- F-BF.1,3a Understanding how changes in the parameters of a function rule correspond to changes in the graph and table of values for the function provides an opportunity to make important connections needed in modeling phenomena. This also ties to students' work with geometric transformations.
- G-CO.1,9-11; G-SRT.1,4 Developing more formal arguments in geometry based on deductive arguments is an important step in more deeply understanding the nature of mathematical evidence.
- G-CO.1,9-11; G-SRT.1,4 Developing more formal arguments in geometry based on deductive arguments is an important step in more deeply understanding the nature of mathematical evidence.
- S-ID.6 Exploring how functions can be used to model a relationship between two quantitative variables not only strengthens students' understanding of finding statistical relationships but also provides an important illustration of the power of functions in explaining relationships.



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Mathematics III Standards Analysis

In the conceptual area of Algebra, students extend their consideration of polynomials, understanding them as a system analogous to the integers and using the structure of an expression to rewrite it in useful ways. They understand the relationship between factors and zeros. They also explore rational expressions. Students reason about and solve a wide range of equations, using graphs or tables of values to approximate solutions, or using inspection, factoring, or other algebraic techniques when appropriate.

Students analyze an increasingly wide range of functions, including polynomial, trigonometric, logarithmic, rational, and other functions. They represent these relationships in different ways and compare functions represented in different ways. They explore and compare key features of these families of functions and express function rules in ways that reveal those features. Students understand logarithm functions as the inverse of exponential functions and can use inverse functions to solve simple equations. As their understanding of modeling grows, students increasingly use unit analysis as a way to understand problems.

In the conceptual area of Geometry, students consider the relationships between two- and three-dimensional objects. Second, students apply geometric concepts in modeling situations.

In the conceptual area of Statistics and Probability, the focus is on inferential statistics. Students understand the role of randomization in statistics and can make inferences and justify conclusions from surveys, experiments, and observational studies.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system that has mathematical coherence, not just as a set of expressions of a specific type. An analogy to the integers can be made (including operations, factoring, etc.). Subsequently, polynomials can be extended to rational expressions, analogous to the rational numbers.
- The understandings that students have developed with linear, exponential, and quadratic functions are extended to considering a much broader range of classes of functions.
- In statistics, students begin to look at the role of randomization in statistical design.

Discussion of Mathematical Practices in Relation to Course Content

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- **Modeling with mathematics (MP.4)** continues to be a particular focus as students see a broader range of functions, including **using appropriate tools strategically (MP.5)**.
- **Constructing viable arguments and critiquing the reasoning of others (MP.3)** continues to be a focus, as does attention to precision, as students are expected to provide increasingly precise arguments.
- As students continue to explore a range of algebraic expressions, including polynomials, they should **look for and make use of structure (MP.7)**.
- Finally, as students solidify the tools they need to continue their study of mathematics, a focus on **making sense of problems and persevering in solving them (MP.1)** is an essential component for their future success.

Fluency Recommendations

- A/F** Students should look at algebraic manipulation as a meaningful enterprise, in which they seek to understand the structure of an expression or equation and use properties to transform them into forms that provide useful information (e.g., features of a function or solutions to an equation). This perspective will help students continue to usefully apply their mathematical knowledge in a range of situations, whether their continued study leads them toward college or career readiness.
- M** Seeing mathematics as a tool to model real world situations should be an underlying perspective in everything students do, including writing algebraic expressions, creating functions, creating geometric models, and understanding statistical relationships. This perspective will help students appreciate the importance of mathematics as they continue their study of it.
- N-Q** In particular, students should recognize that much of mathematics is concerned with understanding quantities and their relationships. They should pick appropriate units for quantities being modeled, using them as a guide to understand a situation, and be attentive to the level of accuracy that is reported in a solution.
- F-BF.1,3** In particular, being able to write a rule to represent a relationship between two quantities is essential to continued meaningful use of algebra. Moreover, students should understand the effects of parameter changes and be able to apply them to create a rule modeling the function.

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Examples of Major Within-Course Dependencies

- Consideration of polynomial expressions naturally interweaves with the exploration of the family of polynomial functions.
- Looking at trigonometric functions requires extending the domain beyond the first quadrant to the unit circle.
- Function composition is needed to understand the idea of inverse functions.

Examples of Opportunities for In-Depth Focus

A-SSE.2,3 Identifying ways to rewrite an expression based on its structure, and doing so intentionally in the context of solving a problem, are habits of mind that can support students in high school mathematics up to and beyond the college and career readiness threshold.

A-APR.1 Considering the polynomials as a mathematical system and extending that perspective to the rational numbers. In particular, connecting the factors of a polynomial to its zeroes is an important consideration in understanding that system.

F-BF.3,4 Extending work with functions and their transformations to additional classes of functions provides a much broader appreciation for functions, how they work, and how they might be used and adjusted to model real-world contexts. Looking at inverses of functions and their use in solving equations could be another facet of this consideration.

G.MG.1-3 Using geometry to model and understand real-world contexts.

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Appendix: Lasting Achievements in K–8

Essay by Jason Zimba from <http://commoncoretools.wordpress.com/2011/06/15/essay-by-jason-zimba-on-pinnacle-standards/>

Most of the K–8 content standards trace explicit steps $A \rightarrow B \rightarrow C$ in a progression. This can sometimes make it seem as if any given standard only exists for the sake of the next one in the progression. There are, however, culminating or capstone standards (I sometimes call them “pinnacles”), most of them in the middle grades, that remain important far beyond the particular grade level in which they appear. This is signaled in the Standards themselves (p. 84):

The evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the boundary defined by (+) symbols in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6–8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume. Because important standards for college and career readiness are distributed across grades and courses, systems for evaluating college and career readiness should reach as far back in the standards as Grades 6–8. It is important to note as well that cut scores or other information generated by assessment systems for college and career readiness should be developed in collaboration with representatives from higher education and workforce development programs, and should be validated by subsequent performance of students in college and the workforce.

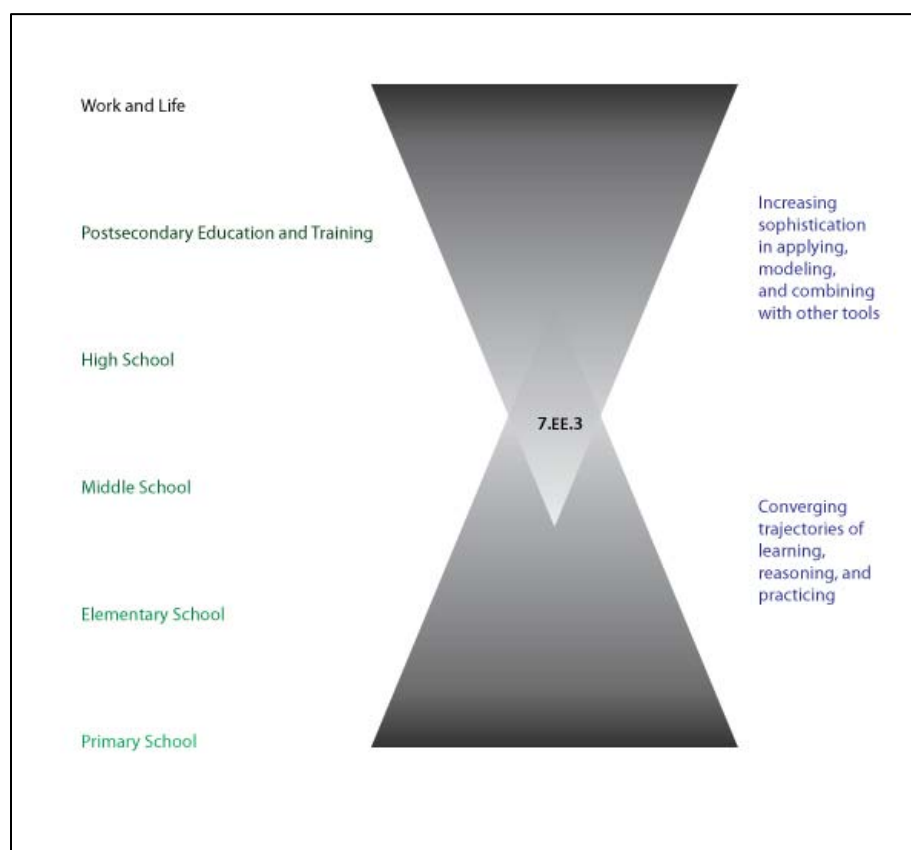
One example of a standard that refers to skills that remain important well beyond middle school is 7.EE.3:

Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.
For example: If a woman making \$25 an hour gets a 10% raise, she will make an additional $\frac{1}{10}$ of her salary an hour, or \$2.50, for a new salary of \$27.50. If you want to place a towel bar $9\frac{3}{4}$ inches long in the center of a door that is $27\frac{1}{2}$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.

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Other lasting achievements from K–8 would include working with proportional relationships and unit rates (6.RP.3; 7.RP.1,2); working with percentages (6.RP.3e; 7.RP.3); and working with area, surface area, and volume (7.G.4,6).

As indicated in the quotation from the Standards, skills like these are crucial tools for college, work and life. They are not meant to gather dust during high school, but are meant to be applied in increasingly flexible ways, for example to meet the high school standards for Modeling. The illustration below shows how these skills fit in with both the learning progressions in the K–8 standards as well as the demands of the high school standards and readiness for careers and a wide range of college majors.



As shown in the figure, standards like 7.EE.3 are best thought of as descriptions of component skills that will be applied flexibly during high school in tandem with others in the course of modeling tasks and other substantial applications. This aligns with the demands of postsecondary education for careers and for a wide range of college majors. Thus, when high school students work with these skills in high school, they are not working below grade level; nor are they reviewing. Applying securely held mathematics to open-ended problems and applications is a *higher-order* skill valued by colleges and employers alike.



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One reason middle school is a complicated phase in the progression of learning is that the pinnacles are piling up even as the progressions $A \rightarrow B \rightarrow C$ continue onward to the college/career readiness line. One reason we draw attention to lasting achievements here is that their importance for college and career readiness might easily be missed in this overall flow.

Additional Note on Modeling (MP.4)

Modeling is a conceptual category in high school (pp. 72–73) as well as a practice standard (MP.4). The practice standard for modeling reads in part as follows:

In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.

According to this description, ordinary numerical or algebraic word problems can be considered modeling tasks. However, the quoted text also describes an arc across the grades. During middle grades and certainly by high school, tasks with a strong modeling component will have more of the hallmarks that are described on pages 72 and 73 of the standards, such as a need to attend to issues of precision, a need to select relevant variables, engagement in the steps in the modeling cycle, or opportunities to use technology.